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FEBRUARY 2013

bundled-up babies and dangerous ice cream

// extraneous solutions?

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contents

Volume 106, Number 6, February 2013

features

418 **Bundled-Up Babies and Dangerous Ice Cream: Correlation Puzzlers**
Kathleen H. Offenholley
Discovering lurking variables encourages students to think critically about correlation.

424 **Punch Up Algebra with POWs**
Mark Pinkerton and Kathryn G. Shafer
An action research study focuses on the teaching strategies used to facilitate Problems of the Week.

432 **Students' Exploratory Thinking about a Nonroutine Calculus Task**
Keith Nabb
An open-ended calculus problem spurs novelties in student thinking and shows the benefits of timely teacher intervention.

446 **Intellectual Engagement and Other Principles of Mathematics Instruction**
Blake E. Peterson, Douglas L. Corey, Benjamin M. Lewis, and Jared Bukarau
What can American teachers learn about high-quality mathematics instruction from the Japanese teacher education process?

452 **Making Sense of Extraneous Solutions**
Jeremy S. Zelkowski
Do you always have to check your answers when solving a radical equation?

on the cover

How could ice cream possibly be dangerous? And what could it have to do with bundled-up babies? Kathleen H. Offenholley uses both in an introductory statistics class to teach students about lurking variables. Want to know how? Turn to page 418.

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The *Mathematics Teacher* (ISSN 0025-5769) (IPM 1213431) is published monthly except June and July, with a combined December/January issue, by the National Council of Teachers of Mathematics at 1906 Association Drive, Reston, VA 20191-1502. Periodicals postage paid at Herndon, Virginia, and at additional mailing addresses. POSTMASTER: Send address changes to *Mathematics Teacher*, 1906 Association Drive, Reston, VA 20191-1502. Telephone: (703) 620-9840; orders: (800) 235-7566; fax: (703) 476-2970; e-mail: nctm@nctm.org. © 2013. The National Council of Teachers of Mathematics, Inc. Printed in the U.S.A.

departments

404 **Reader Reflections**

408 **Sound Off!**
Mario, Monkey Man, Fish, and Bam-Bam: Creative Terminology in Today's Classrooms
Sherry L. Bair and Edward S. Mooney

410 **Media Clips**
The Fiancée Formula // Chocolatefinger
Shaonan (Tony) Gong // Adam Lavallee
Edited by Louis Lim and Lionel Garrison

414 **Mathematical Lens**
How Much Can You Bench?
Chris A. Bolognese
Edited by Ron Lancaster and Brigitte Bentele

440 **March 2013 Calendar and Solutions**
Edited by Margaret Coffey and Art Kalish

459 **Activities for Students**
Dealing Cards with Confidence
Michael Dempsey
Edited by Ruth Dover and Patrick Harless

464 **Connecting Research to Teaching**
Classroom Collaboration: Moving beyond Helping
Corey Webel
Edited by Margaret Kinzel and Laurie Cavey

468 **Technology Tips**
Using FluidMath to Explore Recursive and Explicit Functions
Maria L. Hernandez and Nils Ahbel
Edited by Larry Ottman and James Kett

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472 **Delving Deeper**
Chordic vs. CORDIC: How Calculators and Students Compute Sines and Cosines
Bryan C. Dorner
Edited by Maurice Burke, J. Kevin Colligan, Maria Fung, and Jeffrey J. Wanko

479 **For Your Information**
Publications

480 **The Back Page: My Favorite Lesson**
The Chinese Number Pyramid Puzzle
Sonali Raje, Gail Kaplan, and Michael Krach
Edited by Jennifer Wexler

call for manuscripts

451 **2014 Focus Issue: What Makes Ideas Stick?**

more4U

Look online for these additional items:

- "Punch Up Algebra with POWs" (p. 424): Journal entries
- "Students' Exploratory Thinking about a Nonroutine Calculus Task" (p. 432): An additional student solution
- Mathematical Lens (p. 414): An additional photograph for your own mathematical questions
- Delving Deeper (p. 472): Additional history about the chordic and CORDIC algorithms

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in the next issue

Coming in March 2013:

- "Visible Thinking in High School Mathematics," by Emily Sliman
- "Equivalence Relations across the Secondary School Curriculum," by Lisa Berger
- "Why Is the Midpoint an Average?" by Lingguo Bu
- "Truth and Beauty: Mathematics in Literature," by Marion D. Cohen
- "Sorting Out Ideas about Function," by Amy F. Hillen and LuAnn Malik

GAMES IN THE CLASSROOM

Do you want to get your students excited about mathematics class? The use of games within a classroom serves as a motivational tool, especially for at-risk students, and promotes healthy competition among all.

I constructed a game that I call Mathopoly, a spin-off of the Hasbro game Monopoly®. Every week students earn the opportunity to roll a die to move around the game board to win prizes. Students earn rolls in a variety of ways: by completing all homework assignments for the week, improving a score from the previous test, or being present every day for a three-week period. However, the guidelines for earning rolls should be tailored to your own classroom, and everyone should be able to qualify.

The Mathopoly game board is similar to the Monopoly board. Each property has a mathematical name, such as Addition Alley and Discriminant Drive. I replace the four railroads with Pencil Place and Eraser Place; when landing here, students win a pencil or eraser, respectively. Community Chest and Chance are renamed School Lunchbox and Backpack; students draw a card from a stack and earn a reward such as "move ahead five spaces." When landing on Free Ride, students earn five bonus points; and for Go Directly to Detention, students stay until rolling doubles. Of course, students earn a prize when passing Go.

In addition to being a motivational game, Mathopoly can also be used for cooperative review sessions. Groups win a roll and advance around the board each time they answer a question correctly. Further, the board can be used as an instructional tool. From the start of the game, I have students keep a log of the spaces they land on and then use this log for a lesson on theoretical and experimental probability. A sample question is, "From the class data, what is the probability of landing on an orange space?"

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Clemson, SC, Oct. 16, 2011

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PROBLEM 4, NOVEMBER 2011 CALENDAR

Problem 4 in the November 2011 Calendar (*MT* Oct. 2011, vol. 105, no. 3, p. 200) was stated as follows:

> The sides of a parallelogram measure 2 and 6. The shorter diagonal measures 6. Find the length of the longer diagonal.

One obvious generalization of this problem is to determine how the lengths of the sides of a parallelogram and the lengths of its diagonals relate to each other: The sum of the squares of the lengths of the diagonals is twice the sum of the squares of its dimensions. The nontrigonometric proof supplied in *MT*'s solution is easily adapted to this generalization. Using the diagram from that solution, we replace the width, 2, by w; the length, 6, by l; and the half diagonal, 3, by r. We retain the same use of the symbols x, y, and h (see **fig. 1 [Siegel]**). Application of the Pythagorean theorem three times produces $h^2 + y^2 = r^2$; $h^2 + (x - y)^2 = w^2$; and $h^2 + (x + y)^2 = l^2$. By solving the first equation for h^2 and substituting the result into both the second and third, we obtain the following two equations: $x^2 - 2xy + r^2 = w^2$ and $x^2 + 2xy + r^2 = l^2$. By adding these two equations and then doubling, we obtain the desired result: $2(w^2 + l^2) = (2x)^2 + (2r)^2$.

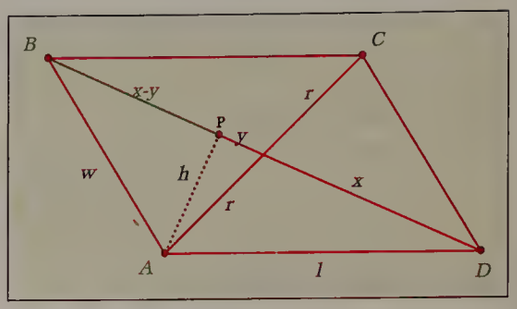

Fig. 1 (Siegel)

An easier proof is available to anyone who knows the law of cosines. If we name one interior angle θ, then the other is $\pi - \theta$ where $\cos(\pi - \theta) = -\cos\theta$. We label the side lengths of the parallelogram l and w and the diagonals d and D. Then, applying the law of cosines twice, we have $D^2 = l^2 + w^2 - 2wl\cos\theta$, and $d^2 = l^2 + w^2 + 2wl\cos\theta$. Adding these two equations completes the proof: $D^2 + d^2 = 2l^2 + 2w^2$.

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THE SLOPE LINK

In their article "A Natural Approach to the Number *e*" (*MT* Feb. 2012, vol. 105, no. 6, pp. 432–39), Helen M. Doerr, Donna J. Meehan, and AnnMarie H. O'Neil suggest several approaches to introducing the mysteries of the number *e* to precalculus students. I liked the graphing activity so much that I decided to try it. My students really enjoyed their search for the "perfect" base that would result in the exponential function, $f(x) = b^x$, whose value is exactly equal to its slope, using the expression

$$\frac{f(x+h) - f(x)}{h}.$$

I found that using $h = 0.0001$ resulted in a closer approximation of *e* for the base. I have always felt that the

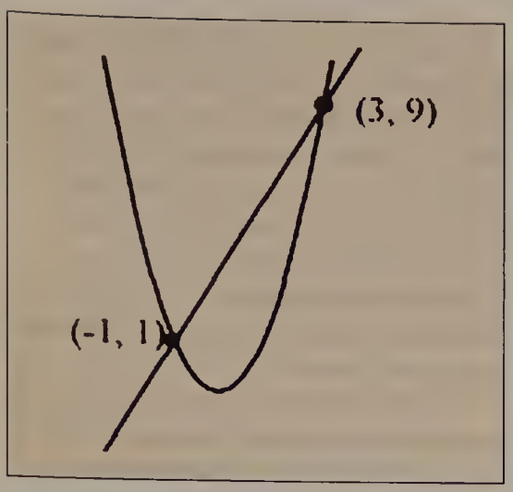

Fig. 1 (Preston)

traditional method of introducing e by calculating the amount earned by investing a dollar at 100 percent interest is a bit contrived. My students, always good sports, humor me—even perking up when, for example, we count how many seconds are in a year to find earnings if interest is compounded by the second.

So I thank Doerr, Meehan, and O'Neil for their innovative method. I would also like to provide another "preview of coming attractions" for precalculus students.

Most textbooks, as they lead up to the concept of average rate of change, ask students to find

$$\frac{f(x+h)-f(x)}{h}$$

for a function f. Students struggle to manipulate the expression correctly, and since the problem is given completely out of context, its purpose is lost on them. Providing a connection between the familiar slope formula in function notation,

$$\frac{\Delta y}{\Delta x}=\frac{f(x_1)-f(x_2)}{x_1-x_2}, \qquad (1)$$

and the difference quotient early on will help students gain insight into the concept of limits and finding instantaneous rates of change in calculus.

Take the function $f(x)=x^2$. To find the average rate of change from $x_1=-1$ to $x_2=3$, we use (1) to find $\Delta y/\Delta x=2$. In other words, even though the function decreases before it increases over $[-1, 3]$, the net result is that for every 1 unit of change in x, the y-value increases 2 units (see **fig. 1 [Preston]**).

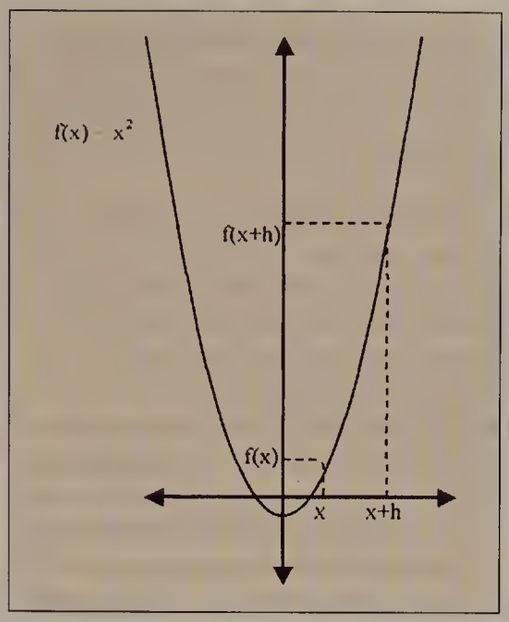

Fig. 2 (Preston)

Alternatively, suppose that we use (1) and let $\Delta x=h$ to find a general expression for the average rate of change in $f(x)$ over the interval $[x, x+h]$ (see **fig. 2 [Preston]**). Substituting into our formula and simplifying, we get the following:

$$\begin{aligned}\frac{\Delta y}{\Delta x}&=\frac{(x^2+2xh+h^2)-(x^2)}{h}\\&=\frac{2xh+h^2}{h}\\&=\frac{h(2x+h)}{h}\\&=2x+h\end{aligned}$$

So the general formula for the average rate of change of the function $f(x)=x^2$ over a domain interval $[x, x+h]$ is given by $\Delta y/\Delta x=2x+h$.

Now, this is where the new approach comes in. Let's use this formula to find the average rate of change over the interval $[-1, 3]$ that we used earlier. Let $x=-1$, the lower boundary of the interval, and let h represent the distance between the lower and upper boundaries of the interval: $3-(-1)=4$. Since we have found a formula for $\Delta y/\Delta x$ in general, we can simply substitute -1 for x and 4 for h, yielding $\Delta y/\Delta x=2(-1)+4=2$, the same result we obtained before.

Finding the average rate of change over an interval has been reduced to a matter of simple substitution, allowing students to make the connection between the two formulas. Emphasizing this relationship in precalculus will enhance students' later understanding of how the definition of the derivative relates to the instantaneous rate of change at a particular point and the slope of the tangent at that point.

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EVELYN'S DISCOVERY

Evelyn Yi Jia Xiang, one of my grade 12 students, proved an interesting result while studying the power law of logarithms. I asked the class to evaluate $\log_4 8$, and Evelyn almost immediately provided the answer of 3/2. When I asked her how she did it, she explained by using a theorem that she proved almost instantaneously. Here is the proof of the theorem:

$$\begin{aligned}\log_{a^n} b^m &= A\\ b^m &= \left(a^n\right)^A\\ \log_a\left(b^m\right)=\log_a\left(a^n\right)^A&=\log_a\left(a^{nA}\right)\\ m\log_a(b)&=nA\\ \frac{m}{n}\log_a(b)&=A\\ \therefore \frac{m}{n}\log_a(b)&=\log_{a^n} b^m\end{aligned}$$

Subsequently, we used the result to evaluate some expressions, such as this one:

$$\begin{aligned}\log_8 1024&=\log_{2^3} 4^5\\&=\frac{5}{3}\log_2 4\\&=\frac{5}{3}\log_2 2^2\\&=\frac{10}{3}\end{aligned}$$

Then we came up with a second proof using the change-of-base formula:

$$\begin{aligned}\log_{a^n} b^m&=\frac{\ln b^m}{\ln a^n}=\frac{m\ln b}{n\ln a}\\&=\frac{m}{n}\cdot\frac{\ln b}{\ln a}=\frac{m}{n}\ln_a b\end{aligned}$$

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ABHINAV'S DISCOVERY

If we find the determinant of

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}$$

using the standard method of expansion by cofactors, we can find a formula that involves twenty-four distinct products. Using some factoring, however, we can simplify this formula even further to a series of just six products involving determinants of 2×2 matrices within the 4×4 matrix. We first expand along the top row and then simplify the 3×3 determinants (see **fig. 1 [Venigalla]**).

What we have found is an expression for the determinant of a 4×4 matrix. This simplification involves six products of 2×2 matrix determinants. For example, if you look at our standard matrix,

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix},$$

the first of our six products, $(af - be) \cdot (kp - lo)$ is just the determinant of the upper-left 2×2 matrix multiplied by the determinant of the bottom-right 2×2 matrix.

Similarly, each of the six products can be seen as a simple multiplication of two 2×2 determinants in the matrix itself. In fact, these six products even form a pattern. The first four products of 2×2 determinants are *added* together (see **fig. 2 [Venigalla]**). The last two products are *subtracted* (see **fig. 3 [Venigalla]**).

Using this "six products" method, one can look at any 4×4 matrix and quickly write a simplified expression for the determinant that does not require lengthy arithmetic or elementary row operations.

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A TRIO OF FACTORABLE QUADRATIC EQUATIONS

On a recent exam, I asked my students to solve the equation $\sqrt[3]{x^3 + 26} = x + 2$.

After cubing both sides and subtracting x^3 from each, students obtained a quadratic equation that can be solved by factoring over the integers.

One student wrote $\sqrt[3]{x^3 + 26x} = x + 2$ by mistake, yet he was still able to solve his resulting quadratic equation by factoring. Curious, I wondered whether $\sqrt[3]{x^3 + 26x^2} = x + 2$ also leads to a quadratic equation solvable by factoring. It does. The equation $\sqrt[3]{x^3 + 26x^3} = x + 2$ easily resolves to $x = 1$. Each of these three equations has the number 1 in the solution set.

From the first three equations emerged the quadratic equations $3x^2 + 6x + 4 = 13$, $3x^2 + 6x + 4 = 13x$, and $3x^2 + 6x + 4 = 13x^2$, all solvable by factoring. A teacher writing problems may find this approach useful for writing similar problems, all of which are solvable by factoring. I sought a method for creating such a nice trio of equations.

In general, the three equations $ax^2 + bx + c = d$, $ax^2 + bx + c = dx$, and $ax^2 + bx + c = dx^2$ can be solved by factoring, so the three expressions $ax^2 + bx + (c - d)$, $ax^2 + (b - d)x + c$, and $(a - d)x^2 + bx + c$ must be factorable. Since $x = 1$ is a solution to the equations, $a + b + c = d$. Replacing d with $a + b + c$ in the expressions, we have

$ax^2 + bx + c - a - b - c$,
$ax^2 + (b - a - b - c)x + c$, and
$(a - a - b - c)x^2 + bx + c$,

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = a\begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b\begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c\begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d\begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

$$a(fkp + gln + hjo - hkn - gjp - flo) - b(ekp + glm + hio - hkm - gip - elo) + c(ejp + flm + hin - hjm - fip - eln) - d(ejo + fkm + gin - gjm - fio - ekn)$$

$$= afkp + agln + ahjo - ahkn - agjp - aflo - bekp - bglm - bhio + bhkm + bgip + belo + cejp + cflm + chin - chjm - cfip - celn - dejo - dfkm - dgin + dgjm + dfio + dekn$$

$$= (afkp - aflo - bekp + belo) + (chin - chjm - dgin + dgjm) + (bgip - bglm - cfip + cflm) + (ahjo - ahkn - dejo + dekn) - (agjp - agln - cejp + celn) - (bhio - bhkm - dfio + dfkm)$$

$$= (af - be)(kp - lo) + (ch - dg)(in - jm) + (bg - cf)(ip - lm) + (ah - de)(jo - kn) - (ag - ce)(jp - ln) - (bh - df)(io - km)$$

Fig. 1 (Venigalla)

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} \quad \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} \quad \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} \quad \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}$$

$$\begin{vmatrix} a & b \\ e & f \end{vmatrix} \cdot \begin{vmatrix} k & l \\ o & p \end{vmatrix} + \begin{vmatrix} c & d \\ g & h \end{vmatrix} \cdot \begin{vmatrix} i & j \\ m & n \end{vmatrix} + \begin{vmatrix} b & c \\ f & g \end{vmatrix} \cdot \begin{vmatrix} i & l \\ m & p \end{vmatrix} + \begin{vmatrix} a & d \\ e & h \end{vmatrix} \cdot \begin{vmatrix} j & k \\ n & o \end{vmatrix}$$

$$(af - be)(kp - lo) + (ch - dg)(in - jm) + (bg - cf)(ip - lm) + (ah - de)(jo - kn)$$

Fig. 2 (Venigalla)

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} \quad \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}$$

$$- \begin{vmatrix} a & c \\ e & g \end{vmatrix} \cdot \begin{vmatrix} j & l \\ n & p \end{vmatrix} - \begin{vmatrix} b & d \\ f & h \end{vmatrix} \cdot \begin{vmatrix} i & k \\ m & o \end{vmatrix}$$

$$- (ag - ce)(jp - ln) - (bh - df)(io - km)$$

Fig. 3 (Venigalla)

which simplify to

$$ax^2 + bx + (-a - b),$$
$$ax^2 + (-a - c)x + c, \text{ and}$$
$$(-b - c)x^2 + bx + c.$$

The first expression requires that $a(-a - b)$ have two factors that add to b. Rewriting $a(-a - b)$ as $-a(a + b)$, we find that $-a + (a + b) = b$. The second expression requires that ac have two factors that add to $-a - c$. Rewriting ac as $(-a) \cdot (-c)$, we find that $(-a) + (-c) = -a - c$. The third expression requires that $c(-b - c)$ have two factors that add to b. Rewriting $c(-b - c)$ as $-c(b + c)$, we find that $-c + (b + c) = b$.

Thus, we need only select d to be the sum of a, b, and c, and we have our desired set of equations. The three equations and their solutions follow:

$ax^2 + bx + c = d$	$\{1, (-a - b)/a\}$
$ax^2 + bx + c = dx$	$\{1, c/a\}$
$ax^2 + bx + c = dx^2$	$\{1, -c/(b + c)\}$

Some words of caution are in order concerning the selection of the values of a, b, c and d:

- If $c = d$, then the first equation will have solution set $\{0, -b/a\}$.
- If $b = d$, then the second equation will not be factorable over the integers unless a is a factor of c. The solution will be

$$\left\{\pm\sqrt{\frac{-c}{a}}\right\}$$

  and, hence, not real when a and c are both positive or both negative.
- If $b = -c$, then $a = d$, and the third equation will not be quadratic.
- If we select values of a, b, and c such that their sum is 0, then $d = 0$, and the trio reduces to one quadratic equation that is solvable by factoring, with one factor necessarily $x - 1$.

If we observe the caveats above, we have an efficient way of writing factorable quadratic equations, with the condition that 1 is always one of the solutions.

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Honolulu, HI, Dec. 2, 2011

Example 1	Mental Steps	Example 2	Mental Steps
$34 \cdot 76 = 2584$	$70 \cdot 34 = 2380$	$36 \cdot 74 = 2664$	$70 \cdot 36 = 2520$
	$6 \cdot 34 = 204$		$4 \cdot 36 = 144$
	$2380 + 204 = 2584$		$2520 + 144 = 2664$
$35 \cdot 78 = 2730$	$35 \cdot 80 = 2800$	$38 \cdot 75 = 2850$	$40 \cdot 75 = 3000$
	$2 \cdot 35 = 70$		$2 \cdot 75 = 150$
	$2800 - 70 = 2730$		$3000 - 150 = 2850$

Table 1 (Call)

MENTAL MULTIPLICATION

If you like to multiply two two-digit numbers mentally, this exercise may interest you. You will discover something very interesting if you exchange the units digits. Check the examples in **table 1 (Call)** and see whether you can discover how the pairs of products, done mentally, are related and how you can obtain one from the other. Question: How can the difference in the products be computed not by subtraction of the products but by playing with the difference of the tens digits and the difference of the units digits?

In **table 1 (Call)**, note that in the first example, $36 \cdot 74$ is larger than $34 \cdot 76$ with $6 > 4$ and $7 > 3$. Then $10(6 - 4) \cdot (7 - 3) = 80$, and $2664 - 2584 = 80$. And in the second example, $38 \cdot 75$ is larger than $35 \cdot 78$ with $8 > 5$ and $7 > 3$. Then $10(8 - 5) \cdot (7 - 3) = 120$, and $2850 - 2730 = 120$. It seems that the product that contains the larger of the two possible smaller factors will be the larger of the two products.

We can use algebra to prove this conjecture, letting T and t represent the respective tens digits of the numbers and letting U and u the respective units digits. Then $(10T + U)(10t + u) = 100Tt + Uu + 10(Tu + tU)$, and $(10T + u) \cdot (10t + U) = 100Tt + Uu + 10(tu + TU)$. We can always assume that one of the products has $U > u$ and $t > T$. Then, $U - u > 0$ and $t - T > 0$, meaning that $(U - u) \cdot (t - T) > 0$. Hence, $Ut - UT - ut + uT > 0$ and $Tu + Ut > TU + ut$ and $10(Tu + Ut) > 10(TU + ut)$. Thus, $(10T + U)(10t + u) > (10T + u)(10t + U)$. The difference in the products is $10(Tu + Ut) - 10(tu + TU)$.

We rewrite $10(Tu + Ut) > 10 \cdot (TU + ut)$ as $10[Tu + Ut - TU - ut] > 0$, which becomes $10[U(t - T) - u(t - T)] > 0$, or $10[(t - T)(U - u)] > 0$; $10[(t - T) \cdot (U - u)]$ is the difference of the products, as conjectured:

Example 1: $U = 6$, $u = 4$, $T = 3$, $t = 7$
Example 2: $U = 8$, $u = 5$, $T = 3$, $t = 7$

This process shows that no matter what product you start with, when the unit digits are switched, the larger product is always obtained from whichever product contains the smaller factor with the larger unit's digit. The tens digit difference does not decide which product is larger; it does contribute to the difference in the products. This approach gives you the option of computing the product that is easier to do mentally and adjusting, if necessary, to obtain the product you really want.

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Fort Ann, NY, May 14, 2012

DO YOU HAVE SOMETHING TO ADD?

Share with readers and the Editorial Panel your opinions about any of the articles or departments appearing in this issue by writing to Reader Reflections. Letters should be sent to mt@nctm.org.

SOUND off!

Sherry L. Bair and Edward S. Mooney

Mario, Monkey Man, Fish, and Bam-Bam:

Creative Terminology in Today's Classrooms

Mathematical precision means more than accuracy in computation or procedures; it also means precision in language. The Common Core State Standards for Mathematics states, "Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning" (CCSSI 2010, p. 7). In our recent experience in working with teachers and students, we have noticed a trend toward teachers using informal, and often creative, language and terminology in an effort to connect with students and make mathematical procedures easier to remember.

Several examples of this trend that we have encountered include use of the following terms to express a mathematical procedure: *Mario, Monkey Man, fish*, and *bam-bam* (see **fig. 1**). These terms themselves carry no mathematical meaning, but in specific classrooms teachers and students have used each in the following ways.

Mario was used when multiplying a monomial by a binomial (e.g., expanding $4(x + 2)$ to $4x + 8$). The teacher and students would simply say, "You Mario it," meaning that the 4 "jumps" to both the x and the 2. This is an attempt to connect with students' familiarity with the Mario character in a popular video game series who jumps from spot to spot. When asked, the teacher said that she believed that using this term helped her connect to students culturally.

In another classroom, *Monkey Man* was used to supplement the term FOIL when multiplying two binomials. Students could "see" that they had completed the entire process when the picture of Monkey Man emerged. The teacher even suggested that students exaggerate the shape of the first and last parentheses so that they could better see Monkey Man's ears.

Fish was used in yet another class to help students remember the procedure of cross multiplication to solve proportions and give specific direction to what quantities to multiply first. Students were to start at the upper right "tail," draw the fish, and follow the line to complete their computations.

Finally, *bam-bam* was used to help students remember to change subtraction into adding the opposite. Changing both signs on a chalkboard or whiteboard quickly produced a bam-bam. (Since hearing this terminology, we learned from another mathematics educator that she has heard students and teachers using *boom-boom* to convey the same idea.) Teachers in this school claimed that students found *bam-bam* easier to remember than changing both signs or changing subtraction into adding the opposite.

THE BEST INTENTIONS CAN GO AWRY

Teachers use these ideas with the best intentions. They are usually trying to connect with students and make the mathematical procedures easier to remember, and most truly believe they are helping their students. So what are the consequences of using these creative terms that the teachers may not be considering?

First, a student or teacher not present in this particular classroom would not know what these terms mean. When the students hear the teacher use these terms and are allowed to use them regularly, they often turn them into verbs that they believe are acceptable mathematical language to represent the procedures or reasoning. For example, a college algebra student may say "I foiled it," when discussing the multiplication of two binomials. Most college-level mathematics teachers are familiar with FOIL and would understand what this student was saying, but we doubt that many college-level mathematics teachers would understand what students were doing when

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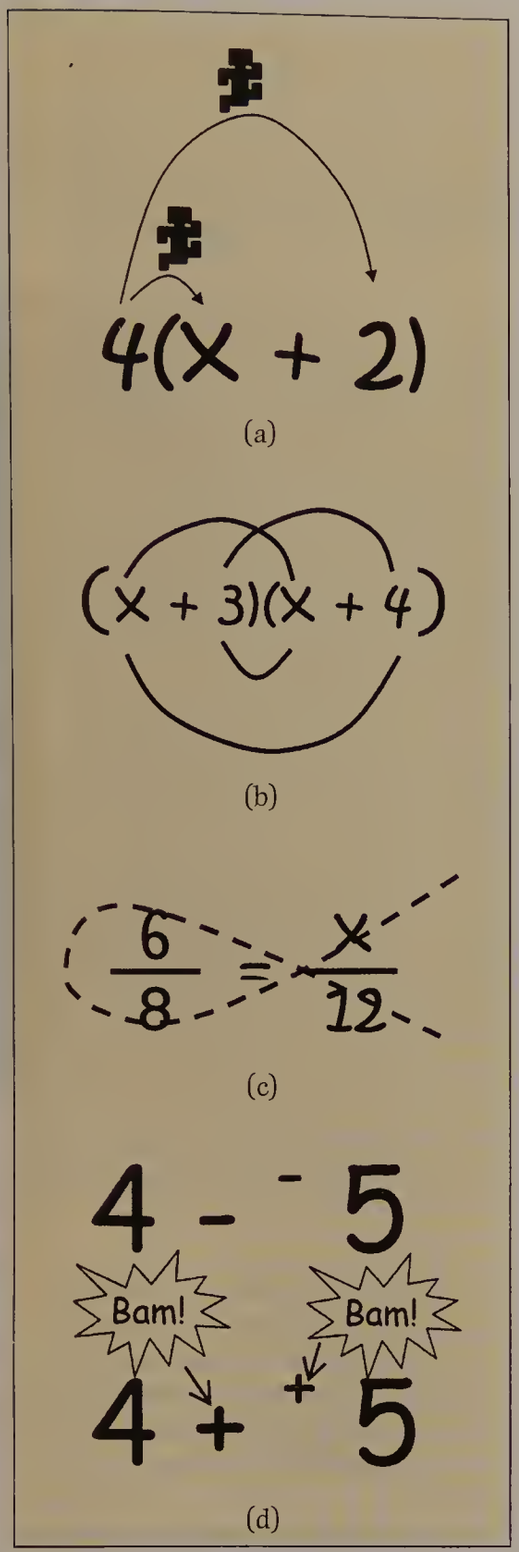

Fig. 1 Some creative terms applied to mathematical procedures include *Mario* (**a**), *Monkey Man* (**b**), *fish* (**c**), and *bam-bam* (**d**).

they say "I made my Monkey Man" or "I Mario-ed it" or "I bam-bammed it." Even if students know what they are doing mathematically, their explanation is not going to be clear to others.

Second, and probably more important, students lose their focus on the actual underlying mathematics of the situation and instead focus on procedures that get them a result that the teacher will accept. For example, most students will not see that Mario, Monkey Man, and FOIL are all mathematical forms of the distributive property. Consider the following sequence of expressions:

$$4(x + 2)$$
$$(x + 2)(x + 6)$$
$$(x + 3)(x^2 + 2x - 1)$$
$$(1 + \sqrt{2} + 3i)(4 + \sqrt{3} - 2i)$$

When students use Mario to expand the first expression and use FOIL or create Monkey Man to expand the second, they need a third new rule for expanding the third expression; the fourth requires yet another procedure or rule. The lack of precise terminology has masked the underlying mathematics, and students do not see the repeated reasoning or the similarities in structure that the CCSSI's standards of mathematical practice require. In fact, students may get to the third expression without knowing that they have been using the distributive property at all.

Let's consider another situation in which these creative terms could hinder understanding of the distributive property in simplification. The expression $4 - (6 - 3x)$ is an example in which students often struggle with distributing the negative. Students who were presented the bam-bam method may try to use this method in one of several incorrect ways, such as these:

- The expression $4 - (6 - 3x)$ becomes two *bam-bam* steps from left to right to create $4 + {}^{-}6 + {}^{-}3x$. These students basically ignore the parentheses completely.
- The expression $4 - (6 - 3x) = 4 - (6 + {}^{-}3x)$ with the first *bam-bam* and "doing inside the parentheses first" and then attempting to *bam-bam* it again to get $4 + ({}^{-}6 + {}^{-}3x)$, only changing two signs rather than distributing. These students think that they have not ignored the parentheses, since they did the inside first, but in reality they still have not distributed the negative sign.

In either case, students have interpreted *bam-bam* to mean that we change only two signs at a time. The lack of understanding of the underlying mathematics results in students making incorrect generalizations and trying to apply mnemonics where they are not appropriate.

USE APPROPRIATE MATHEMATICAL TERMINOLOGY

We have discussed only a few problems in using creative terminology for the distributive property, but we are sure that these examples represent only a small percentage of the creative terminology being used in mathematics classrooms across the country. Our examples were designed to begin a dialogue among mathematics teachers and to help them think beyond the current content that they are teaching to the longer-term implications of using creative mnemonic devices designed to help students remember procedures. If, instead, teachers consistently use appropriate mathematical terminology across grade levels, students will begin to see the regularity in repeated reasoning, and the underlying structure of the mathematics may become clearer.

In addition, when teachers use and encourage the use of correct mathematical terminology in their classrooms, they will help students meet several of CCSSI's standards of mathematical practice, including attending to precision; looking for and making use of structure; and looking for and expressing regularity in repeated reasoning.

REFERENCE

Common Core State Standards Initiative (CCSSI). 2010. Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf.

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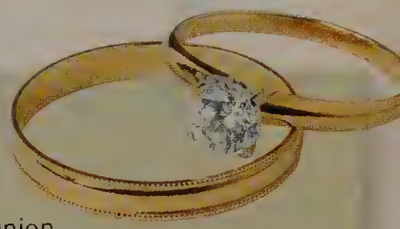
MEDIA clips

Tony Gong // Adam Lavallee

The Fiancée Formula



... In what may qualify as the least romantic study on romance ever done, Australian researchers have popped out a math equation that purports to tell men the best time to propose.

"Although probability isn't the most romantic basis for a marriage, the formula does seem to fit a lot of couples—whether through accident or design," Prof. Anthony Dooley of the University of New South Wales said in a release.

The full version of the so-called "fiancee formula" contains enough letters to stock a season's worth of "Wheel of Fortune." But Dooley and study author Prof. Bruce Brown have boiled it down for a horde of vacillating men:

Take the oldest age at which you'd consider getting married. Call this *n*.

Determine the youngest age at which you'd like to marry. Call this *p*.

Subtract *p* from *n*. Multiply the result by 0.368. (Why 0.368? God knows. This is where the incomprehensible math comes in.)

Add the result to your minimum age. That tells you when you should propose.

For example: [N (39) − P (20)] • 0.368 = 6.992 + 20 = 27 years old.

Dooley and Brown suggest that if the man in the example above wants to maximize his statistical chances at a successful union, he shouldn't propose to anyone until he's 27. However, if he finds himself at age 27 and unmarried, he should immediately propose to the first decent prospect that wanders by.

The mathematicians call the whole exercise "tongue in cheek," although they do claim that using this model will result in a 37 percent chance of optimal marital success.

... In statistical circles, the "fiancee equation" has its roots in something called the "theory of optimal stopping." That theory is most often used in clinical trials....

Source: Cathal Kelly, "Men Advised to Do Fiancée Formula to Determine Best Time to Propose," *Toronto Star*, March 4, 2010, THE CANADIAN PRESS, http://www.thestar.com/living/article/773730--want-to-propose-first-do-the-fiancee-formula

1. Let n denote the oldest age and p the youngest age at which a person would consider getting married. Let y denote the ideal age to propose to a person. Write a mathematical equation for the fiancée formula.

2. In your own words, describe the logic behind the fiancée formula. In particular, what role does the number 0.368 play in the formula?

3. Suppose Bob considers getting married between the ages of 25 and 42. What does the formula give as the ideal age at which he should propose to his girlfriend?

4. Use the formula to calculate your ideal age to propose. Assume that this formula works for both men and women.

5. (*a*) James thinks that his ideal age to propose to his girlfriend is 27 and that his minimum age for getting married is 19. What age would James consider the latest for him to get married?

   (*b*) Jeff believes that he should get married no later than at age 35 and that his ideal age for proposing is 24. What is the earliest age at which Jeff would consider getting married?

6. (*a*) At one point, the article suggests that 27 is the optimal age to get married. Given this information, write an equation in terms of n and p, the latest and earliest age one would consider getting married. What type of relation does this equation represent? Explain.

   (*b*) Graph the relation found in problem 6(a). Round any calculation to three decimal places.

   (*c*) State a reasonable domain and range for this relation.

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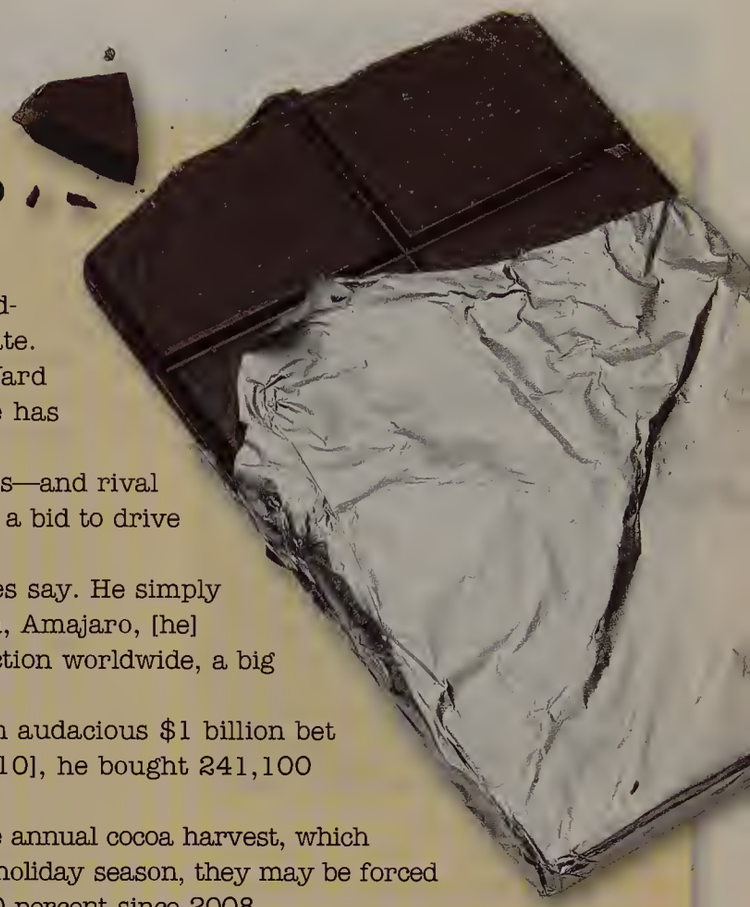
Chocolatefinger



To some, he is a real-life Willy Wonka. To others, he is a Bond-style villain bent on taking over the world's supply of chocolate.

In a stroke, a hedge fund manager here named Anthony Ward has all but cornered the market in cocoa. By one estimate, he has bought enough to make more than five billion chocolate bars.

Chocolate lovers here are crying into their Cadbury wrappers—and rival traders are crying foul, saying Mr. Ward is stockpiling cocoa in a bid to drive up already high prices so he can sell later at a big profit.

Mr. Ward, 50, is not some rabid chocoholic, former employees say. He simply has a head for cocoa. And, through his private investment firm, Amajaro, [he] now controls a cache equal to 7 percent of annual cocoa production worldwide, a big enough chunk to sway prices.

Now, traders here are buzzing that Mr. Ward has placed an audacious $1 billion bet in the London market for cocoa futures. This month [July 2010], he bought 241,100 metric tons of beans, they say....

The fear is that Mr. Ward will become the go-to source until the annual cocoa harvest, which starts in October. With candy makers starting to stock up for the holiday season, they may be forced to pay him ever-higher prices—and cocoa has already jumped 150 percent since 2008.

Chocolate lovers should not worry too much, analysts said. Cocoa accounts for only about 10 percent of the price of most ordinary chocolate bars.

Source: Julia Werdigier and Julie Creswell, "Trader's Cocoa Binge Wraps Up Chocolate Market," http://www.nytimes.com/2010/07/25/business/global/25chocolate.html?_r=1&scp=2&sq=supply%20price&st=cse

1. After buying 241,100 metric tons of cocoa beans in July 2010, Ward controls 7% of annual cocoa production. Using these figures, calculate the total annual cocoa bean production.

2. The article mentions that Ward has bought enough cocoa beans "to make more than five billion chocolate bars." A chocolate bar weighs 44 grams; it can contain as little as 1% true cocoa and as much as 11%.

   (*a*) How much cocoa (in grams) is in one chocolate bar if each bar contains 1% cocoa?

   (*b*) How much cocoa (in grams) is in one chocolate bar if each bar contains 11% cocoa? (This percentage is the standard in the United States.)

3. Assuming that Ward can make 5 billion chocolate bars each containing between 1% and 11% cocoa, what is the minimum and maximum amount of cocoa he needs to produce these chocolate bars? Give your answer in metric tons (1,000,000 grams are in a metric ton).

4. Cocoa beans are used for items other than chocolate bars. What percent of Ward's 241,100-metric-ton purchase will be used for the 5 billion chocolate bars? Use the upper bound from your answer to problem 3 (the United States requires chocolate bars to consist of 11% cocoa).

5. According to the article, the price of cocoa "jumped" 150% between 2008 and 2010.

   (*a*) Using the Internet or some other source, check the accuracy of this statement.

   (*b*) Determine the average annual rate of increase over those two years.

6. By purchasing a large amount of cocoa, Ward is betting that prices will continue to rise. If the price of cocoa continues to increase at the average annual rate determined in the answer to problem 5, by how much has his $1 billion investment increased?

7. Since "cocoa accounts for only about 10 percent of the price of most ordinary chocolate bars," by approximately how much would the price of a chocolate bar have increased? Have prices truly reflected this change?

MEDIA CLIPS solutions

"The Fiancée Formula" answers

1. The equation for the fiancée formula can be written as $y = p + 0.368(n - p)$.

2. Answer may vary, but here is a typical response. The $(n - p)$ part of the formula represents the range (or window) of years that one considers for getting married. Multiplying this quantity by 0.368 results in a specific point in this window at which a person would like to get married. Therefore, the expression $p + 0.368 \cdot (n - p)$ is equivalent to saying, "Your optimal age for proposal is about 36.8% through the range of years you would consider getting married, starting from the earliest age you would consider getting married."

3. Using the formula, we get $y = 25 + 0.368(42 - 25) = 31.256$. Therefore, Bob should propose at about age 31.

4. Answers will vary, but the process is the same as in answer 3.

5. (*a*) Using the formula $y = p + 0.368 \cdot (n - p)$, substitute $y = 27$, $p = 19$. Rearrange the equation to solve for n: $n \approx 40.74$. Therefore, the approximate latest age at which James would consider getting married is age 41.

   (*b*) Using the formula $y = p + 0.368 \cdot (n - p)$, substitute $y = 24$, $n = 35$. Rearrange the equation to solve for p: $p \approx 17.59$. Thus, 18 is approximately the earliest age at which Jeff would consider getting married.

6. (*a*) If $y = 27$, we get $27 = p + 0.368 \cdot (n - p)$. This expression represents a linear relationship because there are only two variables in this equation and the degree of each variable is 1.

   (*b*) Before we can graph this relationship, we need to isolate one of the variables:

$$\begin{aligned} 27 &= p + 0.368(n - p) \\ 27 &= p + 0.368n - 0.368p \\ 27 - 0.368n &= p - 0.368p \\ -0.368n &= 0.632p - 27 \\ n &= -1.717p + 73.370 \end{aligned}$$

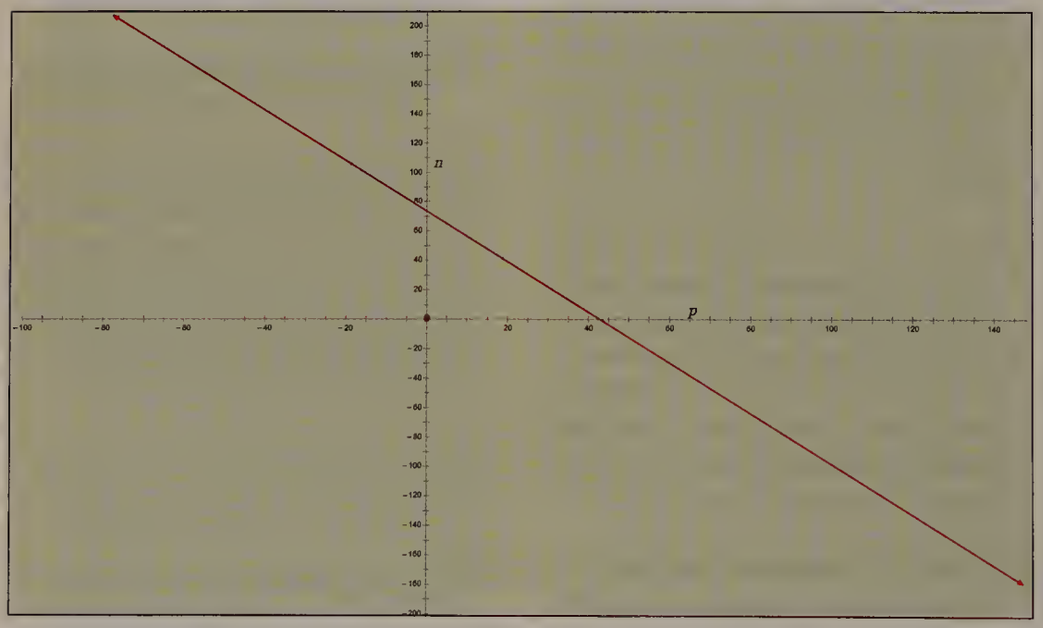


Fig. 1 ("The Fiancée Formula")

   For the graph of this equation, see **fig. 1 ("The Fiancée Formula")**.

   (*c*) The parts of the line to the left of the n-axis and below the p-axis are not realistic because a person's age cannot have a negative value. In other words, to make the relation more realistic, we can restrict the domain and the range as follows:

   Domain:
   $\{p \mid 0 \le p \le 42.732,\ p \in R\}$

   Range:
   $\{n \mid 0 \le n \le 73.370,\ n \in R\}$

"Chocolatefinger" answers

1. To calculate the total annual cocoa bean production, we need to determine the total production that 241,100 is 7% of. A proportion leads to the equation $0.07x = 241{,}100$. Thus, about 3.4 million tons of cocoa are produced annually. This number agrees with global production reports.

2. (*a*) If a chocolate bar weighs 44 grams and 1% of its weight is cocoa, then it would contain 0.44 grams of cocoa: $0.01 \cdot 44 = 0.44$.

   (*b*) If 11% of the bar is cocoa, then each bar would contain 4.84 grams of cocoa: $0.11 \cdot 44 = 4.84$.

3. To make 5 billion chocolate bars with 1% cocoa per bar, or 0.44 grams per bar, Ward would need $5{,}000{,}000{,}000 \cdot 0.44 = 2{,}200{,}000{,}000$ grams, or $2{,}200{,}000{,}000/1{,}000{,}000 = 2{,}200$ metric tons of cocoa beans. To make 5 billion chocolate bars with 11% cocoa, or 4.84 grams per bar, he would need $5{,}000{,}000{,}000 \cdot 4.84 = 24{,}200{,}000{,}000$ grams, or $24{,}200{,}000{,}000/1{,}000{,}000 = 24{,}200$ metric tons of beans. So, to make 5 billion chocolate bars, Ward would

need between 2,200 and 24,200 metric tons of beans.

4. If the total amount of cocoa is 241,100 metric tons and 24,200 metric tons are used for chocolate bars, then $24{,}200/241{,}100 \approx 0.100$, or 10%, of the total purchase will be used for chocolate bars.

5. (*a*) The world cocoa price in January 2008 was \$2220 per metric ton. The price increased to \$3520 in January 2010. This change represents a percent increase closer to 160%, not 150% (see http://www.indexmundi.com/commodities/?commodity=cocoa beans&months=60).

   (*b*) The world cocoa price in 2010 is about 1.586 times the price in 2008. Let $(1 + r)^2 = 1.586$, where r represents the annual percent increase. Solving for r yields an annual interest rate of about 25.9%.

6. In February 2013, it will be 3 years and 7 months (or 3.583 years) since Ward's purchase. Using 25.9% for the annual rate, we find that the billion-dollar investment will be worth $1 \cdot (1.259)^{3.583}$ dollars, or approximately \$2.28 billion (more than double the original investment).

7. According to the results above, the world cocoa price increased $(1.259)^{3.583}$, or about 228%. This increase affects only 11% of the cost of the chocolate bar, so we would expect the cost of the bar to rise by 11% of 228%, or about 25% altogether. Has this happened?

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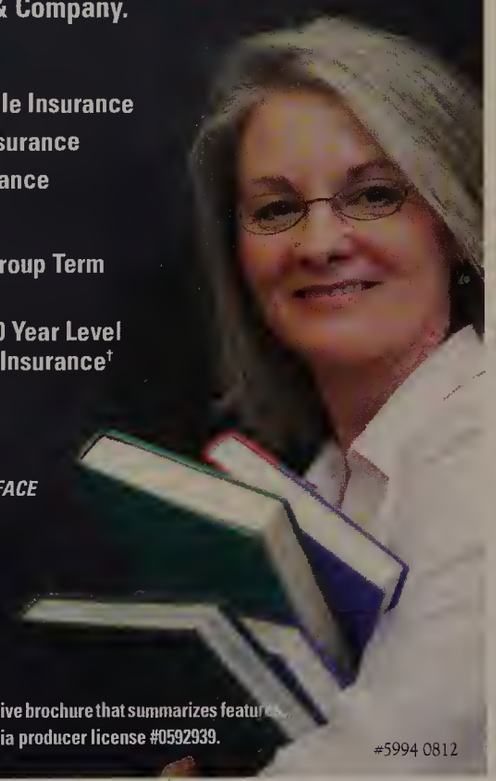

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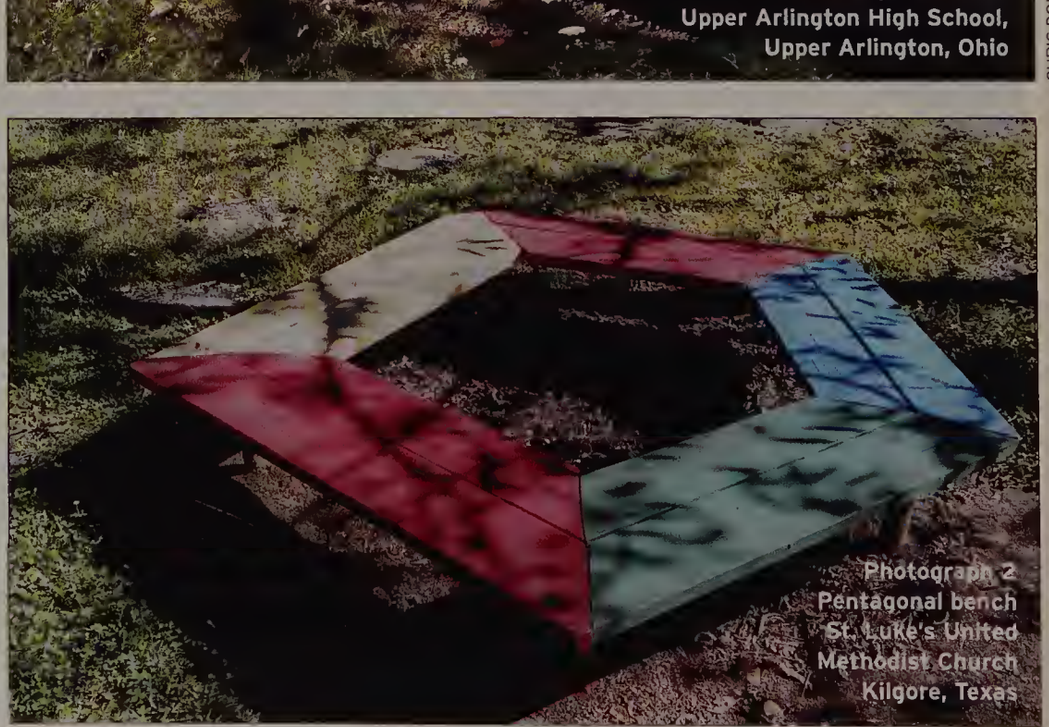
Chris A. Bolognese

How Much Can You Bench?

Photograph 1
Hexagonal bench
Upper Arlington High School,
Upper Arlington, Ohio

CHRIS BOLOGNESE



Photograph 2
Pentagonal bench
St. Luke's United
Methodist Church
Kilgore, Texas

RON LANCASTER

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The hexagonal bench outside Upper Arlington High School in Upper Arlington, Ohio, a northwest suburb of Columbus, has some interesting mathematical connections. Take a seat—and explore geometry, trigonometry, transformations, combinatorics, and series in these problems.

1. The bench in **photograph 1** is made of two regular, concentric hexagons.

   (*a*) What geometric relationship exists between the two hexagons?

   (*b*) Suppose that the outer hexagon has area 30 ft.^{2} and the inner hexagon has area 20 ft.^{2}. If the perimeter of the outer hexagon is approximately 20.3 ft., find the perimeter of the inner hexagon.

   (*c*) Assume that the diagram of the bench shown in **figure 1** is made of six congruent isosceles trapezoids with base lengths a and b and leg length c. Suppose that $b =$ 5 ft. and $a = 4$ ft. Find the total area of the bench. Determine this area in as many ways as you can.

   (*d*) Determine the total area of the hexagonal bench explicitly as a function of a and b.

   (*e*) Suppose that the sum of the lengths of a and b is 8 feet. What is the value of c if the total area of the hexagonal bench is 41.3 ft.^{2}?

   (*f*) **Figure 2** shows a diagram of a hexagonal bench constructed around a preexisting tree so that the circumference of the tree trunk is 14.8 ft. The distance between the bases of a seat of the bench is to be 1.5 ft. What would be the area of this hexagonal bench if the tree is inscribed within the bench?

2. A bench of a similar design is shown in **photograph 2**. This pentagonal bench is found outside St. Luke's United Methodist Church in Kilgore, Texas. **Figure 3** shows an idealized pentagonal bench.

   (*a*) The inner pentagon of the bench is an example of a dilation of the outer pentagon. Imagine dilating a regular pentagon about its center iteratively with a ratio of similitude of 2/3 (see **fig. 4**). If the sum of all the areas of the pentagon is 36 ft.^{2}, what is the area of the outermost pentagon?

   (*b*) Apply the same dilation to the hexagonal bench, with the ratio of similitude of m/n where $m < n$ and $m, n > 0$. If the sum of all the hexagons' areas is K ft.^{2}, what is the area of the outermost hexagon in terms of m, n, and K?

   (*c*) Suppose that a group of students wishes to sit around either the hexagonal bench shown in **photograph 1** or the pentagonal bench shown in **photograph 2**. If each student sits on a different trapezoidal seat, how many more distinct seating arrangements are there on the hexagonal bench than on the pentagonal bench?

3. Suppose that a bench were to be made in the shape of a regular heptagon instead of a hexagon.

   (*a*) Design a new heptagonal bench that has the same area as the hexagonal bench in problem 1(*c*).

   (*b*) What is the locus of points (a, b) for all possible heptagonal benches satisfying the requirements in problem 3(*a*)?

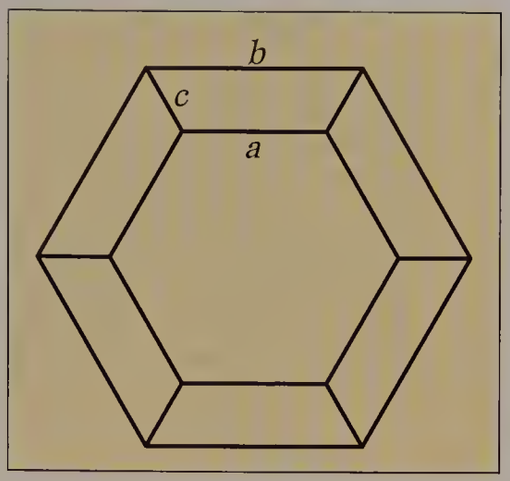

Fig. 1 The idealized hexagonal bench consists of two regular, concentric hexagons.

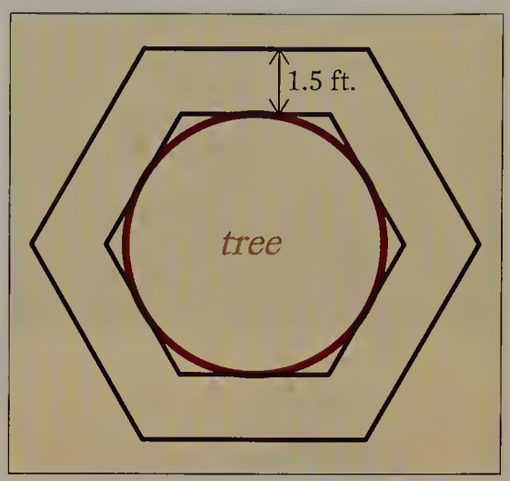

Fig. 2 The tree is tangent to each side of the hexagonal bench.

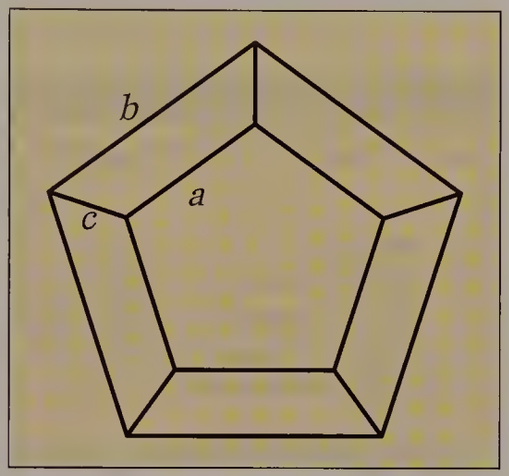

Fig. 3 The idealized pentagonal bench consists of two regular, concentric pentagons.

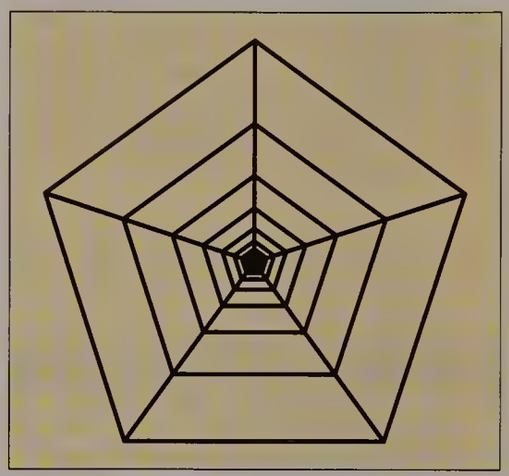

Fig. 4 A pentagonal fractal can be formed with a ratio of 2/3 between successive iterations.

MATHEMATICAL LENS solutions

1. (*a*) Two geometric figures are said to be *similar* when they have the same shape but not necessarily the same size. Such shapes have proportional corresponding sides and congruent corresponding angles.

(*b*) The ratio of the area is 30 : 20 = 1.5; this is the square of the scale factor, which we will call k. Then $k^2 = 1.5$, so k is approximately 1.2247. Since the ratio of the perimeters of two similar figures is equal to the scale factor, we have $20.3/P \approx 1.2247$, where P is the perimeter of the inner hexagon. Thus, P is approximately 16.58 ft., or about 16 ft. 7 in.

(*c*) There are many ways to approach this problem. Here are three:

Method 1

The formula for the area of a trapezoid is $A = (a + b)h/2$ for base lengths a and b and height h. The base angles of each trapezoid in **figure 5** are 60°. Using equilateral triangles, we find that the leg length c is 5 – 4 = 1 ft. To find the height h of just one seat of the bench, we use the Pythagorean theorem:

$$h = \sqrt{1^2 - 0.5^2} \approx 0.866 \text{ ft.},$$

or ratios of the sides of a 30-60-90° triangle. Thus, $A = (4 + 5) \cdot (0.866)/2 \approx 3.897$ ft.^{2}, so the total area is 6(3.897), or approximately 23.38 ft.^{2}.

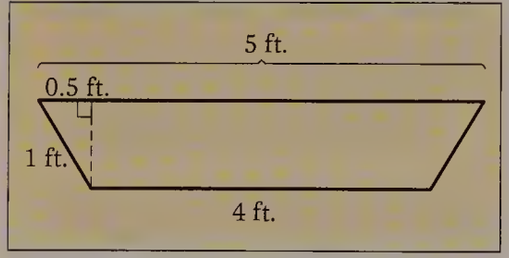

Fig. 5 The area of the trapezoid can be determined if the base angles are known.

Method 2

The linear scale factor is $k = 5/4$; thus, the ratio of areas is $k^2 = 1.5625$. If we let A be the area of the inner hexagon, then the required area is $1.5625A - A = 0.5625A$. We can find the area of the inner hexagon using the formula

$$\text{Area} = 6\left(\frac{s^2\sqrt{3}}{4}\right)$$

because the area of a regular hexagon is six times the area of an equilateral triangle of side length s. Thus, the area is

$$0.5625 \cdot 6\left(\frac{4^2\sqrt{3}}{4}\right) \approx 23.38 \text{ ft.}^2.$$

Method 3

Using The Geometer's Sketchpad®, we can construct two regular hexagons to scale (see **fig. 6**). The difference in areas is 23.38 ft.^{2}.

(*d*) The difference in areas of the two hexagons is

$$6\left(\frac{b^2\sqrt{3}}{4} - \frac{a^2\sqrt{3}}{4}\right) = \frac{3\sqrt{3}}{2}\left(b^2 - a^2\right).$$

(*e*) We use the previous result:

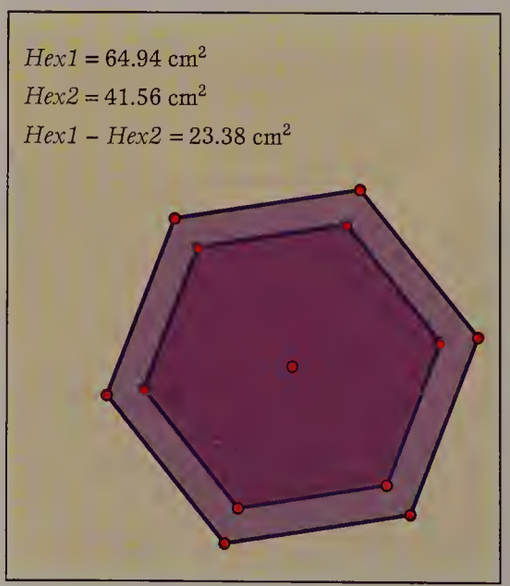

Fig. 6 A technology solution can be accomplished using The Geometer's Sketchpad.

$$A = \frac{3\sqrt{3}}{2}\left(b^2 - a^2\right) = \frac{3\sqrt{3}}{2}(b + a)(b - a)$$

$$= \frac{3\sqrt{3}}{2}(8)(b - a) \approx 41.3 \rightarrow$$

$$b - a = \frac{41.3}{12\sqrt{3}} \approx 1.987$$

If we drop an altitude from a vertex of the trapezoid's smaller base, we create a 30-60-90° triangle with hypotenuse c and with a leg of length $(b - a)/2$ opposite the 30° angle. Thus, $\sin 30° = (b - a)/2c$, so $c = (b - a)/(2\sin 30°) = b - a \approx 1.987$ ft. Curiously, the distance c is precisely equal to the difference of bases in such a hexagonal bench.

(*f*) The radius of the region containing the tree trunk is $14.8/(2\pi) \approx 2.355$ ft. Using 30-60-90° triangles, we find that the side length of the inner triangle is $2(2.355)/\sqrt{3} \approx 2.719$ ft. and the side length of the outer triangle is $2(3.855)/\sqrt{3} \approx 4.451$ ft. Using the formula from part (*d*), we find that the total area is

$$\frac{3\sqrt{3}}{2}(4.451^2 - 2.719^2) \approx 32.26 \text{ ft.}^2.$$

2. (*a*) If the ratio of similitude is 2/3, then the ratio of the areas is 4/9. The sum of an infinite geometric series is $a/(1 - r)$ where a is the first term and r is the common ratio, provided that $|r| < 1$. If A is the outermost area, then we have the following:

$$A + \frac{4}{9}A + \left(\frac{4}{9}\right)^2 A + \cdots = \frac{A}{1 - \frac{4}{9}} = \frac{9}{5}A \rightarrow$$

$$\frac{9}{5}A = 36 \rightarrow A = 20 \text{ ft.}^2$$

(*b*) Note that the ratio of successive areas is m^2/n^2; hence,

$$A + A\left(\frac{m^2}{n^2}\right) + A\left(\frac{m^2}{n^2}\right)^2 + \cdots = K.$$

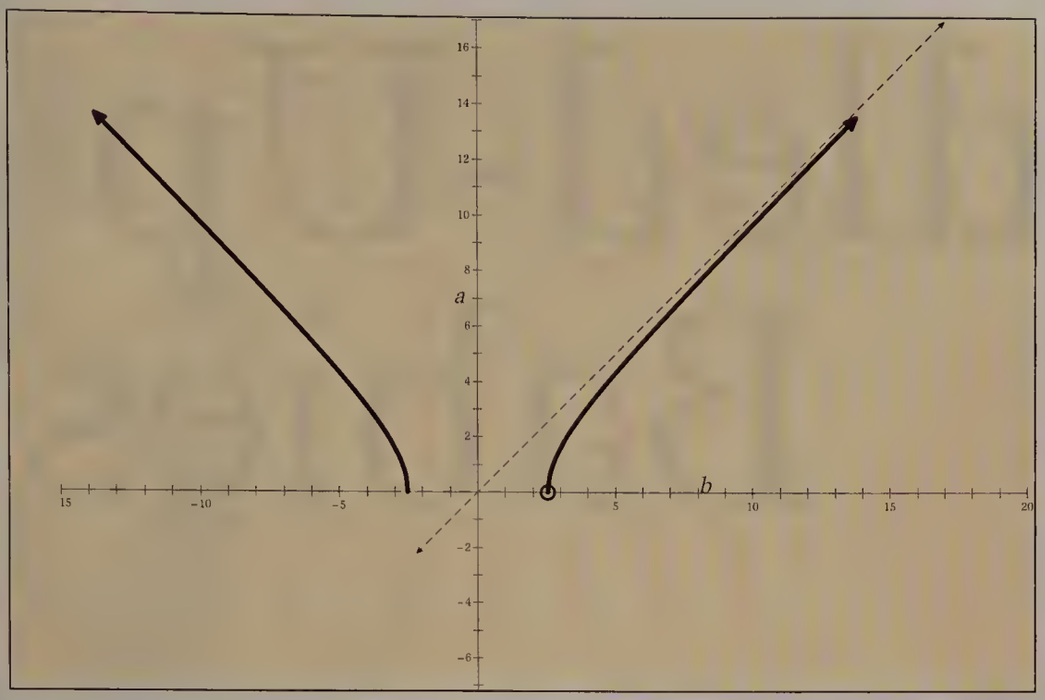

Fig. 7 The asymptote $a = b$ emerges as b approaches infinity.

Factoring out A and applying the sum of a converging geometric series with $r = m^2/n^2 < 1$, we arrive at the following:

$$A \cdot \frac{1}{1-\frac{m^2}{n^2}} = K \rightarrow$$

$$A \cdot \frac{n^2}{n^2-m^2} = K \rightarrow$$

$$A = K \cdot \frac{n^2-m^2}{n^2}$$

Note that this result can be written as

$$A = K \cdot \left(1-\left(\frac{m}{n}\right)^2\right),$$

if we take into account the original scale factor m/n.

(*c*) To solve this problem we need to use circular permutations. For n objects to be placed around a circle, there are $n!/n = (n-1)!$ possible arrangements. Thus, the hexagonal bench offers $5! - 4! = 4! \cdot (5-1) = 4! \cdot 4 = 96$ more arrangements.

3. (*a*) First, we find a formula for the area of a regular heptagon of side length s. The central angle of any regular heptagon is $(360/7)°$. Let

Table 1

Values of b for Selected Values of a

a (ft.)	b (ft.)
0.5	2.5853
1	2.7265
2	3.2302
6	6.5141

$$\theta = \frac{1}{2} \cdot \frac{360°}{7} \approx 25.7143°.$$

Using right-triangle trigonometric ratios, we find that $h = x \cdot \cot\theta$ where h is the apothem and x is half the side length. The area of the regular heptagon is $A = 7 \cdot (1/2) \cdot (2x) \cdot h = 7x^2 \cdot \cot\theta \approx 14.5356x^2$. If we let $s = x/2$, the area is around $3.6339s^2$. If a and b are the side lengths of the inner and outer heptagons, respectively, we need $3.6339(b^2 - a^2) \approx 23.38 \rightarrow b^2 - a^2 \approx 6.4$. By picking various values of a, we can get associated values of b (see **table 1**).

(*b*) The set of all values (a, b) form a *hyperbola*. The domain restrictions on a and b necessitate that we choose points that lie in quadrant I (see **fig. 7**).

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Bundled-Up Babies

Discovering lurking variables encourages students to think critically about correlation.

At the Borough of Manhattan Community College (BMCC) in New York City, I am fortunate to teach a remarkably diverse group of students. BMCC is fourth among all community colleges in awarding degrees to minority students and in awarding degrees to African Americans; the BMCC student body is approximately 37 percent Hispanic, 33 percent black, 15 percent white, and 15 percent Asian. In addition, a significant proportion of our students come from other countries or are first-generation Americans for whom English is their second language. BMCC's students also reflect a wide range of ages; many have returned to school after caring for children or working outside the home (BMCC 2010).

BMCC's diversity makes teaching introductory statistics an exciting opportunity to share insights across cultures and ages. I have students work collaboratively, in groups or pairs, so that they can learn from one another, not only from the teacher. In addition, students find it less intimidating to

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Dangerous Ice Cream: ation Puzzlers

Kathleen H. Offenholley

BABY: GLOW IMAGES; ICE CREAM: GMVOZD/ISTOCKPHOTO

offer their own, evidence-based opinions when they have the support of a peer group. Many of my students come from cultures in which following rules is far more important than thinking independently. Students often tell me that they had never before expressed an opinion in a mathematics class.

One of the most important areas in which critical thinking is essential is in the study of linear correlation. Generally, as Witte and Witte (2010) note, "Two variables are positively related if pairs of scores tend to occupy similar relative positions (high with high and low with low) in their respective distributions, and they are negatively related if pairs of scores tend to occupy dissimilar relative positions (high with low and vice versa) in their respective distributions" (p. 128).

To help my students begin to appreciate this idea, I ask them to suggest two things in their own lives that might have a positive or negative correlation. Students typically provide examples such as grades and hours of studying (positive), exercise and weight (negative), and money and hours

worked (positive). Occasionally, a student gives an example of a negative correlation that uses a more colloquial meaning of the term *negative*—"The more you eat, the more you weigh, and that's bad." Other students quickly jump in with corrections. It is important for students to realize that a negative correlation does not necessarily imply a bad thing nor a positive correlation a good thing.

Students pick these examples with the idea that the change in one variable directly causes the change in the other—for example, more studying causes higher grades. Students often have the idea that all correlation is causal. This initial way of understanding correlation is fine, but I want students to finish the unit with a much deeper understanding, a more nuanced view of what it means for two variables to be correlated.

In addition, the misconception that correlation implies causation is a typical one for students (Liu, Lin, and Tsai 2009) and is probably the most common error made when statistical studies are interpreted in the media (Utts 2003). We all want statistics students to leave class better able to analyze the news without falling prey to such errors of interpretation. As a countermeasure, as this article proposes, students can have fun with correlation puzzlers in which a third factor, a lurking variable, is the cause. Students become detectives, trying to find the answers to the puzzlers. In this way, they improve their ability to analyze statistical studies and deconstruct the information in advertisements.

Sullivan (2010) defines a *lurking variable* as "an explanatory variable that was not considered in a study but that affects the value of the response variable in the study" (p. 17). To help students understand the idea of lurking variables, I ask them, in groups, to find the lurking variable for at least one of five puzzlers I give them. By having five problems to choose from, students feel less pressure and usually come up with answers to at least two, often three, of the five puzzlers.

A description of each puzzler and the students' reactions follow.

PUZZLER 1: FIRE DAMAGE AND FIRE TRUCKS

The amount of damage at the scene of a fire has a positive correlation with the number of fire trucks on the scene. Explain.

This puzzler, the easiest of the five, is commonly used in descriptions of lurking variables. Typically, however, students react with bewilderment. "Does the water from the fire hose damage things?" asks a student in one group. Initially, some groups are completely silent.

I walk from group to group, encouraging students to think about the links between fire trucks and fire damage. Soon, a student from each group is excitedly sharing what he or she has figured out. The lurking variable is the extent of the fire; we can relate this numerically to the number of alarms—a three-alarm, four-alarm, or five-alarm fire. The bigger the fire, the more fire trucks are called in. At the same time, the damage from the fire is more extensive.

Once students understand that the fire trucks do not cause the damage, they are able to think more clearly about correlation. They can then move on to solve one of the other four puzzlers with a better understanding of what to look for.

PUZZLER 2: BUNDLED-UP BABIES AND OUTDOOR TEMPERATURE

The age at which a baby first learns to crawl has a negative correlation with outdoor temperature. Explain.

Students may first have to review the definition of *negative correlation.* In this case, the higher the outdoor temperature, the earlier babies learn to crawl; conversely, the lower the temperature, the later babies learn to crawl.

Some groups find this puzzler too difficult to even begin to solve and skip to the next one. Often one of the older women in the class who has returned to college after having children wants to tackle this puzzler. She will start talking to her group about all the clothes babies have to wear in the winter whenever they go outdoors, and the group begins to realize the connection. If they do not, sometimes a simple hint will suffice: "What is it like for babies in the winter?" In New York City

KICKSTAND/ISTOCKPHOTO

in the winter, the subway is full of babies bundled in pounds of padded winter clothes. My students all agree—those babies are not crawling anywhere!

A pedagogical advantage of this puzzler is that it has more than one possible explanation, and students begin to get a feel for how much conjecture may be involved in identifying a possible lurking variable. The lurking variable may be the amount of outdoor time the babies get (more in the summer, when it is warmer) or how restricted babies are in crawling with the amount of clothing they have to wear in the winter.

In a follow-up to the original article in which this data set can be found (Benson 1993), an author in Ontario, Canada, found that the relationship did not hold in more extreme climates (Bartlett 1998). My students who come from warmer climates, such as Haiti, recognize this as well, and a lively discussion can ensue about how children fare in different climates and cultures.

PUZZLER 3: DANGEROUS ICE CREAM AND DROWNINGS

The number of ice cream servings consumed has a positive correlation with the number of drownings. Explain.

Students' initial reaction to this puzzler is to talk about direct causation. "You are never supposed to eat before you swim," begins one of the group members. The group then develops a chain of causation that sounds like something their mothers told them when they were little: "The person eats too much ice cream, then goes swimming and gets a cramp, and then drowns." Reminders that students should be looking for indirect links, not causation, are usually necessary.

In any case, the idea that eating before swimming causes cramps and drowning is an old, erroneous one (Kohn 1991, pp. 9–10). (See also http://www.snopes.com/oldwives/hourwait.asp for an excellent dissection of this myth.)

At this point, students are still thinking deterministically about a single person who is bound for a sure, swift death because he or she ate ice cream; rather, they should be thinking about large numbers of people, an increase in ice cream sales, an increase in drowning, and so on. It may help to remind students to think about general trends, such as what else is going on when people swim. In most groups, there will be at least one student who realizes that outdoor temperature is the lurking variable. The higher the outdoor temperature, the more ice cream is consumed and, also, the more people who go swimming. Some groups will recognize that the relationship here is similar to that in puzzler 2; many can now go back to puzzler 2 and solve it.

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PUZZLER 4: CAR COST AND GAS MILEAGE

In 2005, there was a negative correlation between car cost and gas mileage. Explain.

This surprising result comes from an analysis I did of more than 200 new cars from the 2005 *Consumer Reports* database. I challenge students to find a connection between the two variables to account for the fact that cars that cost more tended to have lower gas mileage. Because most of my students commute by subway, not by car, this puzzler is hard for them. When I taught in suburban schools, this puzzler was often one of the first that those students were able to solve.

The lurking variables in this puzzler are engine size and car weight. The more expensive cars tend to have larger engines, which get fewer miles per gallon. In addition, the more expensive cars tend to be larger and, on average, less fuel efficient. This is another puzzler in which students can see that there is more than one possible answer and that the answers may be intertwined. Hybrid cars are, of course, outliers in this data set. More recently, with many more expensive small cars on the market, this relationship has not held.

PUZZLER 5: FOOT SIZE AND READING ABILITY

Foot size and reading ability have a positive correlation. Explain.

My students' initial reaction to this puzzler is that it simply cannot be true. Some students hold out their small feet and protest, "But I read well," while the basketball players joke about how their large feet must mean that they are really smart. I urge students to think generally rather than particularly (not about

themselves). A hint I give is that this correlation pertains only to a certain part of the population—only to children, not adults.

The lurking variable is the child's age. The older a child, the better he or she reads. Further, the older a child, the taller he or she becomes and the larger his or her feet grow.

This puzzler has some similarities to the bundled-up babies puzzler, whose answer does not hold in all climates. This puzzler can also provoke further conversation. For example, what would the scatter plot look like for adults?

WHOLE-CLASS DISCUSSION

We finish this section by coming together as a class to discuss all five puzzlers and to write down more formally the lurking variable for each one. I emphasize that we are looking for a numerical quantity—temperature, age, weight—and that we are looking for associations rather than direct causation.

Working on these puzzlers has helped my students get a feel for what real statisticians do—that is, finding links, identifying causes, and generating ideas. Later in the semester, students are much more willing to try to tease out lurking variables and less afraid to suggest possible ideas.

Two studies that we examine later include one from Denmark showing that beer drinkers have lower IQs than wine drinkers (2001, described in *Bandolier: Journal of Evidence-Based Thinking about Health Care*) and a study from the Czech Republic indicating that beer-drinking scientists produce fewer published papers (Yoon 2008). We also look at subway ads for home insurance that claim that home renters are 50 percent more likely than homeowners to be burglarized. These studies are more complicated to analyze than the puzzlers. It is likely that the results from the beer drinkers-vs.-wine drinkers study and the renters-vs.-homeowners study involve class and income as lurking variables. The Czech Republic study, which involves a nonrandom sample of Czech ornithologists, is fun to include, because students get a chance to see what a mess can be made with a selected data set rather than a random sample.

Other possible statistics to examine include any studies in which the size of the population is the lurking variable. For example, there is a positive correlation between the number of murders and the number of schools and between the number of murders and the number of churches.

Through these puzzlers, I draw on both statistical and educational theory to develop a process in which students become their own agents of statistical inquiry in a manner that encourages even the most reticent, across all cultures and ages, in my classroom.

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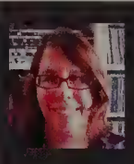

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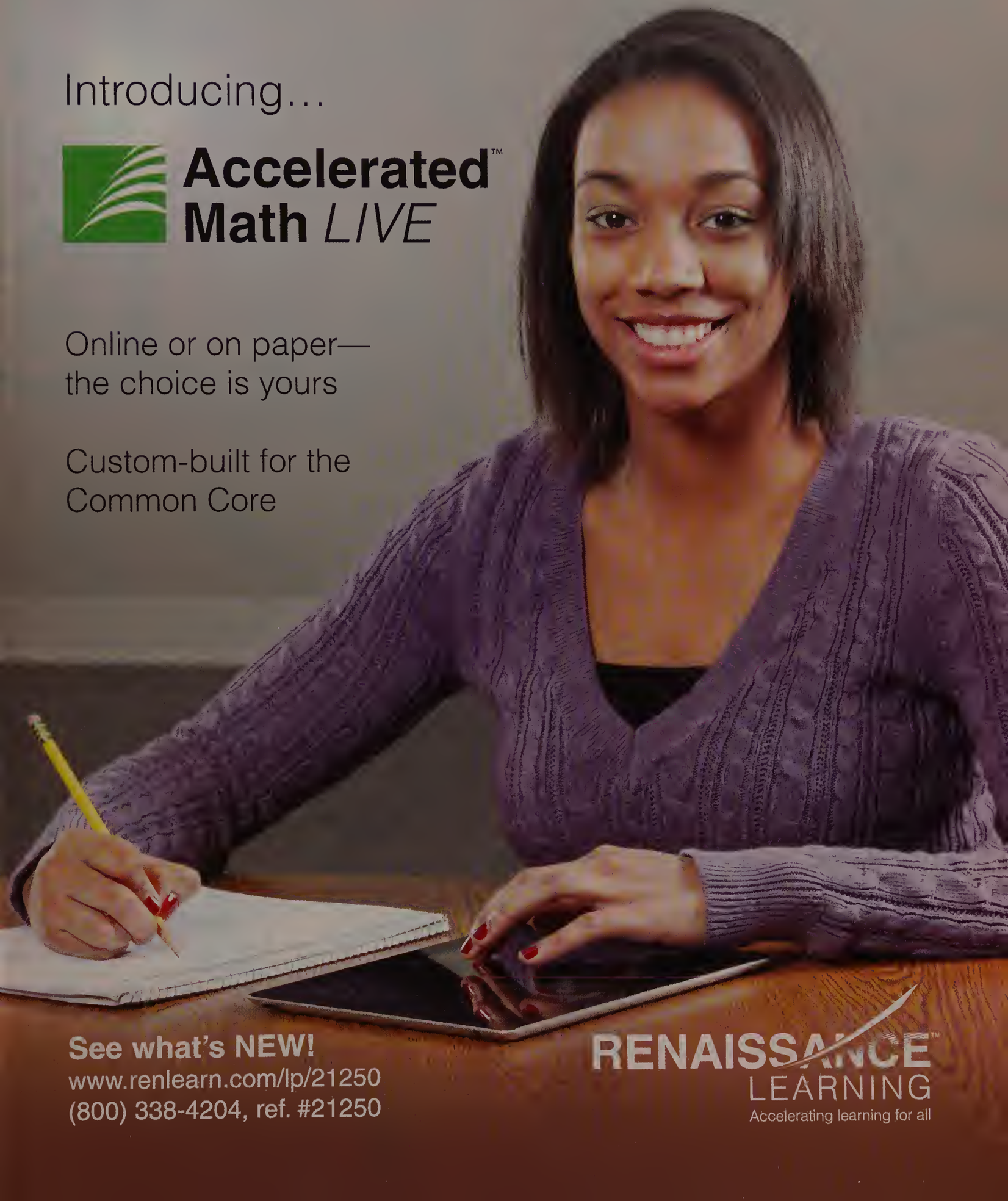

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An action research study focuses on the teaching strategies used to facilitate Problems of the Week.

Mark Pinkerton and Kathryn G. Shafer

Problem solving is a necessary component of developing a strong mathematics curriculum that will help all students achieve their life goals, regardless of their specific academic plans (NCTM 2000). The first Standard for Mathematical Practice (CCSSI 2010, p. 6) declares that students "make sense of problems and persevere in solving them." Teachers frequently ask how to implement a curriculum centered on problem solving within a high-stakes testing environment. How do we as teachers integrate problem solving into today's classroom? Hiebert (2003) simply states that students must be given the opportunity to reason and make sense of the mathematics in a given course.

Hiebert and Wearne (2003) argue that creating this "opportunity" is necessary because problem solving leads to understanding. Because the traditional teaching approach often has tipped the balance toward telling students too quickly how to solve problems, teachers need to correct the balance by thinking again about how to allow students to do more of the mathematical work. Students' understanding depends on it (Hiebert and Wearne 2003, p. 5).

Creating this opportunity is not so simple and requires perseverance, but the investment of time and energy will profoundly affect the experience of both teaching and learning. What day-to-day instructional decisions do teachers need to make if they believe that problem solving is a vehicle for learning mathematical content? Here coauthor Pinkerton describes a yearlong experience designed to provide a sustained problem-solving opportunity for his students.

PINKERTON'S RESEARCH PROJECT

Early in my teaching career, I focused on problem solving and implemented Problems of the Week (POWs). The development and examination of this approach and its effects on students did not begin until I began working through an action research study as part of a graduate degree program. The main goal of my action research was to examine and reflect on my attempt to create a classroom that effectively engaged students in problem solving.

The research took place in a rural Indiana high school during the 2008–9 school year. The students in the study were ninth and tenth graders enrolled in a seventy-minute extended algebra class; these weaker students were given more time to master the first-year algebra curriculum. The extended algebra class structure allowed students to receive extra time and individual attention. This extra time also gave me the opportunity to devote class time to problem solving.

NCH UP
EBRA with
POWs

Over the course of the year, I did the following:

- I modified and refined the teaching strategies that I used to implement problem solving as I actively reflected on my overarching goal of engaging students.
- The rationale for and selection of problems changed as I became more skilled at integrating problem solving into the curriculum and increased my awareness of the cognitive challenge.
- My use of questioning evolved as I learned to answer a question with a question and pose more probing questions.

Selected examples from my action research report will address these three modifications. I also present and discuss student achievement data from local and state assessments. We then summarize our findings.

Teaching Strategies

My problem-solving focus was structured around a problem of the week (POW) using a format designed by Tsuruda (1994). The POWs were assigned on Monday and due on Friday. During the first two weeks of the year, we completed the POWs as a class so that students could learn the four-part solution write-up (see **appendixes A** and **B** for examples of student work showing a problem statement, plan, work, and answer). After we worked through two examples, the goal was for students to complete the majority of the work outside class and turn in the problem write-up on Friday. This homework grade was worth 20 points, 5 points for each section; one POW was worth four daily homework assignments.

I graded POWs over the weekend and returned them early the following week with comments as needed in each section. For example, I commented on strong ideas and solution methods and provided specific feedback for students struggling with the write-up. Feedback was also given through whole-class discussion of eloquent solutions, common solutions, and common errors throughout the year.

Despite working through the first two POWs as a class, students were reluctant to complete the problems on their own. Students complained about the amount of writing and explanation required by the POW format, and only 37 percent completed the third POW. As a result, I implemented weekly problem-solving sessions to help build problem-solving skills and support completion of POWs. The sessions occurred at least once a week and were designed to give students the opportunity to solve problems and engage in discussion of individual and group presentations. Such sessions, lasting thirty to forty-five minutes, were often scheduled on quiz days or the second day of a two-day lesson.

Nails and Pails

A pail containing 40 nails weighs 175 grams. The same pail with 20 nails weighs 95 grams. How much does each pail weigh? How much does each nail weigh?

Fig. 1 The Nails and Pails problem (Collier 2000, p. 19) was assigned during the review for a test covering systems of equations.

In an attempt to create an environment in which students felt comfortable participating in problem solving, I did not grade the work from these sessions. As a result, students were eager to post and present solutions, even students who often failed to complete POWs. The problems used during the sessions reflected current class topics, if possible, and were similar to the problems given in the POWs (see, e.g., **fig. 1**).

During a typical problem-solving session, students worked on two to four problems with a partner or a small group. Students or groups were selected to post solutions on the board. Students who had posted solutions were asked to present and defend their work, and then the class discussed the validity of both the solution and the method. The presentations paralleled the POW process as students shared their plan, process, and justification with the class. The student presentations showcased multiple strategies and were more powerful than a lecture on problem solving. However, when necessary, I presented or illustrated a solution method that students had not suggested.

Early in my study (September and October), I chose students or groups to present only if they had the correct answer. I did not devote any class time to discussion of incorrect solutions. After working through a couple of incorrect solutions with individual groups, I discovered the power of a discussion that results from an incorrect solution. These conversations often provided a chance to explore common misconceptions and look more deeply into the mathematics behind the problem.

As a result of the weekly problem-solving sessions and lots of encouragement, more students were completing POWs and exhibiting an increased level of confidence in mathematics. I noticed that as students became more confident, they often spent more time thinking about the problem before starting work and were not afraid to attempt the problem in multiple ways.

Appendix A (see p. 427) is an example of how Sara (all names are pseudonyms) approached

POW no. 6 with increased confidence. Sara routinely completed POWs, but her methods were usually elementary, and she was often unsure about her work. When reviewing her work on POW no. 6, I thought that she had merely created a hundreds chart and checked each number, a viable but brute-force method. However, after reading her explanation, I noticed that she had not only created the hundreds chart but also used divisibility to eliminate incorrect answers. Her solution demonstrates what I refer to as "thinking about the problem instead of just doing the problem." Sara's willingness to use divisibility and to reason about the problem indicates that she was beginning to engage the mathematics at a higher level of mathematical thinking, consistent with a higher level of confidence.

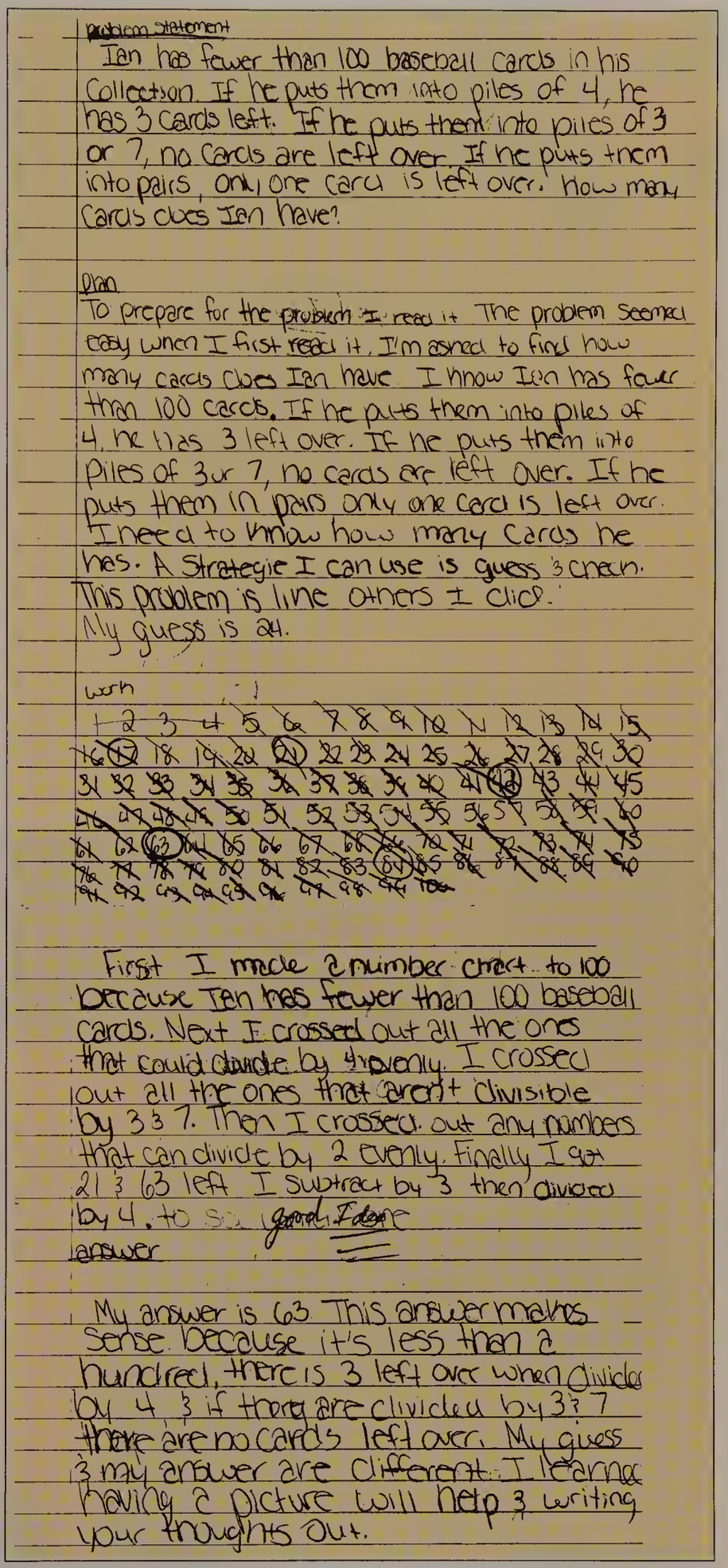
After a month of these problem-solving sessions, the quantity and quality of completed POWs seemed to increase, although a number of students still refused to complete them regularly. It was not clear whether these students were unable to do the work or simply did not want to take the time to complete the assignment. To help students and parents understand the importance of the POWs, I sent a letter to parents explaining the POWs and the importance of problem solving. I also held conferences with parents of students who were not completing POWs and reviewed the material sent home in the parent letter. Although I did see some student improvement after this parent contact, I was still not happy with the number of students who failed to complete the POWs.

One final step taken to encourage—more accurately, force—more students to complete POWs was to assign after-school detentions for incomplete POWs. However, this draconian step made POWs even less popular, and administering the POWs during detention was time-consuming and unsuccessful. As I reflected on my role in this process, I realized that I had very little control over the problem solving that students did outside class.

However, I still believed that the POWs were an important activity for students to complete and for me to grade. As a result, I decided to change the daily class schedule to give students a chance to start the POW in class. Students were given fifteen minutes on either Monday or Tuesday to look at the POW and begin planning or solving the problem. Throughout the week, I reminded students to work on the POW and encouraged them to engage in discussion about the problem when extra time was available.

With the weekly problem-solving sessions in place and class time provided to help students start POWs, I began to focus on the following questions:

- What kind of problems am I selecting?
- What types of questions are students asking?
- How am I responding to the students' questions?


problem statement
Ian has fewer than 100 baseball cards in his collection. If he puts them into piles of 4, he has 3 cards left. If he puts them into piles of 3 or 7, no cards are left over. If he puts them into pairs, only one card is left over. How many cards does Ian have?

plan
To prepare for the problem I read it. The problem seemed easy when I first read it. I'm asked to find how many cards does Ian have. I know Ian has fewer than 100 cards. If he puts them into piles of 4, he has 3 left over. If he puts them into piles of 3 or 7, no cards are left over. If he puts them in pairs only one card is left over. I need to know how many cards he has. A strategie I can use is guess & check. This problem is like others I did. My guess is 24.

work

First I made a number chart to 100 because Ian has fewer than 100 baseball cards. Next I crossed out all the ones that could divide by 4 evenly. I crossed out all the ones that aren't divisible by 3 & 7. Then I crossed out any numbers that can divide by 2 evenly. Finally I got 21 & 63 left I subtract by 3 then divided by 4. to so [illegible] answer

My answer is 63. This answer makes sense because it's less than a hundred, there is 3 left over when divided by 4, & if they are divided by 3 & 7 there are no cards left over. My guess & my answer are different. I learned having a picture will help & writing your thoughts out.

Appendix A Sample student work on POW no. 6 (Baseball Cards) shows the four-part solution.

Problem Statement: A carpenter goes to a hardware store to buy a specific number of pounds of nails for his business. The first store he enters sells his favorite-sized nails at 9 cents per pound. At this price, the carpenter realized that he would be short $7.15 if he tried to buy the nails. The carpenter goes to a competing hardware store and finds that the nails sell for 6 cents a pound. He realizes that after paying for the nails that he would have $2.45 left over. How many pounds of nails is he trying to buy?

Plan: I read the problem and gathered math supplies. The problem seemed hard when I read it. I'm asked to find how many pounds of nails the carpenter is trying to find. I know that his favorite sized nails are 9¢ a pound, and if he tried to buy them he would be $7.15 short. I also know that another type of nail is 6¢ a pound, and if he bought them he'd have $2.45 left. I need to find how many pounds he's trying to buy. I can use system of equations to solve the problem. I don't remember doing any problems like this before. I think the carpenter is trying to buy 10 pounds of nails.

Work: To solve the problem, I made 2 equations with the information in the problem. The first equation ($m-.09p=-7.15$) is the money he had minus 9¢ per pound is equal to −$7.15 because he would've been short $7.15. The second equation, ($m-.06p=2.45$) is the money he had minus 6¢ per pound of nails is equal to $2.45 because he had $2.45 left over. I subtracted one equation from the other and got p = 320 lbs. When I tried the problem first, I didn't change the signs to the opposite. That didn't work. When I tried it again, I changed the sign and it worked. When I got stuck, I asked Nicole for help.

$m-.09p=-7.15$
$-m+.06p=-2.45$
$-.03p=-9.60$
$-.03 \quad -.03$
$p=320$ lbs

Answer: The carpenter wanted to buy 320 pounds of nails. Yes, the answer makes sense. Because if you plug 320 in place for the p in one of the equations, it equals the same. No, there couldn't be other correct answers. My original guess was 10 pounds, when he really wanted to buy 320 pounds. I learned that solving problems are a lot easier if you just think about it.

Appendix B The Buying Nails POW produced this student solution.

Problem Selection

I began the year using questions from *Problem-Driven Math: Applying the Mathematics beyond Solutions, Grades 6–8* (Krulik and Rudnick 2005) and *Menu Collection* (Collier 2000) as well as problems I had compiled during my graduate work. At first, I was picking problems that were below grade level, assuming that the majority of students had the skills to solve them (see **appendix A** for an example). My hope was to build students' confidence in problem solving and the POW write-up process.

As students became more comfortable with the POWs—or at least gained more experience with the process—I began assigning problems that were more appropriate to their grade level. I gave thought to the type of mathematics present in the problem and when to assign it: Problems were chosen to introduce material, coincide with class lessons, or review material previously covered in the course.

The student work shown in **appendix B** relates to a review POW that I used while preparing for the state exit exam. The problem involves a system of equations setup, a common question on the state exam. In this example, Kate was able to set up two equations and used the elimination method. This example also provides evidence of a student realizing that her first attempt contained sign errors and then continuing to work toward a correct solution. I was encouraged to see the self-correction occur and noted that this correction may not have been evident without the problem-solving write-up.

In addition to using grade-level problems and deciding when to assign these, I started searching for problems that could be solved using multiple methods. After a few classroom presentations, I found that problems with multiple-solution methods presented more opportunities for discussion as the students worked on and presented the problems.

During my action research study, I kept a reflective journal of my teaching experience. Excerpts from this journal are provided online (go to **www.nctm.org/mt032**).

Questioning

At the start of the school year, I noticed that students were skilled at asking questions that resulted in my giving them the answer, so I began to answer a question with a question. This approach allowed me to ask students to explain their work or to verbalize the problems they were encountering. Questioning skills took time to develop and required perseverance by both my students and me. This type of questioning took more time because students' questions could not be answered with a number or by responding yes or no. However, this questioning technique—answering a question with a question—helped students improve their

explanations and their approaches to problem solving. The vignette in **figure 2** represents a typical conversation regarding POW no. 23 (Buying Nails).

In addition to answering questions with questions, I focused on asking probing questions during problem-solving sessions. While moving around the room, I tried to listen to conversations and ask students questions about the answer or method being used. At first, I noticed that students immediately assumed that their answer or method was incorrect if it were being questioned. This habit was hard to break; even by the end of the year, a few students still retained this defensive attitude. The probing questions often helped students justify their work and find mistakes in their solution; it also appeared to make students more comfortable and confident when presenting solutions.

This type of questioning usually led to a deeper discussion about the topic or method. A journal reflection captures my thoughts in the first month of the study.

Student Data

I was curious to see how students in the problem-solving study compared with peers who were not part of the study. I used standardized exams that both my students and their peers took during the school year. Students in the study were identified as needing extra help in mathematics and completed 1.5 years' worth of algebra instruction in one year. Their peers were enrolled in the regular algebra classes taught by other instructors. When compared with their peers, the group of students in this study performed very well on the state exit exam and showed growth on the Northwest Educational Assessment (NWEA) mathematics test.

On the Indiana Algebra 1 exit exam in 2009, 60 percent of the students in the study passed, compared with a 29 percent pass rate by other students at the high school and a state pass rate of 41 percent (see **table 1**). State results include data for all students taking algebra, from both middle school and high school. NWEA scores (fall and spring) for ninth graders in the study increased by 6.4 while other ninth graders enrolled in algebra at the high school showed an increase of 3.1. Data from the 2008 NWEA mathematics status norms (at **www.nctm.org/mt**) showed growth of 2.4 for the school year.

WHO INFLUENCED WHOM?

Who was influenced more by Pinkerton's yearlong focus on problem solving in this algebra classroom—the students or their teacher? A problem-solving approach to teaching mathematics is not

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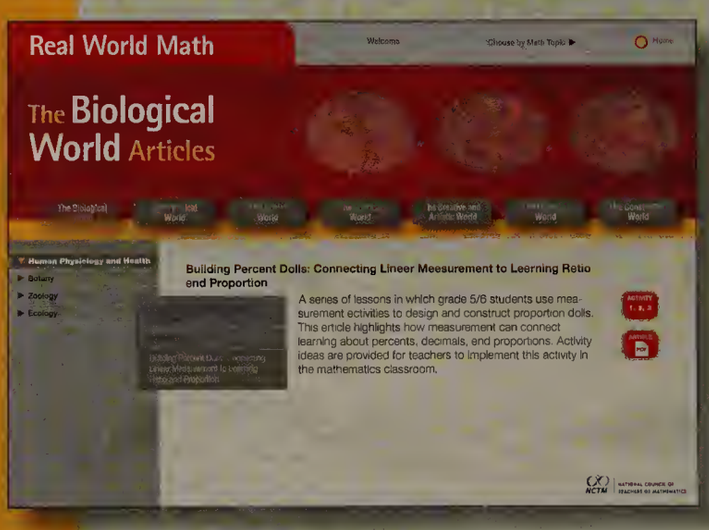
Real World Math
The Biological World Articles
Building Percent Dolls: Connecting Lineer Meesurement to Leerning Retio end Proportion
A series of lessons in which grade 5/6 students use measurement activities to design and construct proportion dolls. This article highlights how measurement can connect learning about percents, decimals, and proportions. Activity ideas are provided for teachers to implement this activity in the mathematics classroom.

Buying Nails (POW no. 23)

A carpenter goes to a hardware store to buy a specific number of pounds of nails for his business. The first store he enters sells his favorite-sized nails at 9 cents per pound. At this price, the carpenter realized that he would be short $7.15 if he tried to buy the nails. The carpenter goes to a competing hardware store and finds that the nails sell for 6 cents per pound. He realizes that after paying for the nails that he would have $2.45 left over. How many pounds of nails is he trying to buy?

Source: http://www.nctm.org/publications/calendar/index.aspx?journal_id=3

Sample Conversation

Student: I am lost. I don't know what to do.
Teacher: I am not sure I understand. What have you tried?
S: Well, I think I can solve this by writing equations.
T: Why?
S: System of equations.
T: Oh, so how many equations do you need?
S: Two equations. But I only need to know pounds.
T: So you don't have another unknown? That might make it hard to write two equations. Have you tried to write equations?
S: Yeah, but they are not working out.
T: Let's take a look. What happens when you go to check out at a store?
S: You pay for your stuff.
T: How?
S: Money?
T: Yes, but how does the transaction take place? Does the cashier just take all of your money regardless of how much you spend?
S: No, he subtracts your cost from the money you give him, and then you get change.
T: OK, write that sentence as an equation.
S: Money – Cost = Change.
T: Can we replace these words with numbers and variables?
S: We know the change and cost for each equation. Would I use a negative for when he is short $7.15?
T: Sounds reasonable. What about the money?
S: I don't know … it doesn't say anything about the money.
T: What do we do when we don't know a number?
S: Use a variable, maybe m.
T: Sounds good. What method do you think you might use to solve the system?
S: I am guessing elimination.

Fig. 2 The Buying Nails problem (POW no. 23) typically generated discussion similar to that reported here.

Table 1 Indiana Algebra 1 Exit Exam Scores

	Number of Students	% Passing Exit Exam	Mean Score*
Students in problem-solving study	30	60	605.9
Peers not in study	95	29	460
State results	91,899	41	526.6

* Range for exit exam: 200–800; passing score: 579

easy and requires perseverance by both teacher and students. But the process became easier as Pinkerton felt more confident in his ability to—

- shift the focus from incomplete POWs to in-class "grade-free" problem solving;
- provide students with opportunities to solve problems;
- pick better problems;
- respond to student questions;
- ask questions of students; and
- facilitate a class discussion.

Since completing the study in 2009, Pinkerton has continued using a problem-solving focus, but his approach and problem selections have changed and evolved with the types of classes and students he teaches. He has not found a perfect set or sequencing of problems to use with a particular class. However, he has noticed that after students become accustomed to problem solving, they often enjoy problems simply for the mathematical challenge and less for the storyline of the problem. There has been an important shift in the source of mathematical authority in the classroom: When students' questions are redirected back to them, students begin to reflect on their solution and solution process instead of immediately looking to the teacher for approval.

Despite refining his teaching strategies during the study, Pinkerton is continually challenged in attempting to focus classes on problem solving. Many teachers are confronted with varying student ability, larger class sizes, shrinking class time, and many more challenges. At times it will be necessary to shorten problems or activities, adjust classroom routines, combine lessons, and be creative as we prepare students for standardized exams and follow the Standards for Mathematical Practice.

Ultimately, this action research study solidified our belief that a mathematics classroom focused on problem solving results in increased student engagement and improved mathematical communication for both students and teacher. We believe that significant emphasis must be given to the classroom discourse that takes place during the problem-solving assignment or session. If we equate good prompts and questions as jabs and punches in the art of mathematical conversation, we cannot expect to win if we are hanging on the ropes.

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For excerpts from the reflective journal that Pinkerton kept during his action research study, download one of the free apps for your smartphone and then scan this tag to access **www.nctm.org/mt032**.

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STUDENTS' EXPLORATIONS

An open-ended calculus problem spurs novelties in student thinking and shows the benefits of timely teacher intervention.

Keith Nabb

Introductory calculus often generates intriguing questions that can ignite an appreciation for the subject of mathematics. Here are a few:

- Under what conditions is $(f(x)g(x))' = f'(x) \cdot g'(x)$? (Dunkels and Persson 1980)
- Why can the chain rule be memorized by "canceling" differentials?
- Do more useful coordinate systems exist that can assist in evaluating challenging integrals? If so, what might these systems look like?

These questions open doors to advanced mathematical thinking and harness many elements of research-oriented mathematics. Such questions also offer greater incentives for students to think and reflect. For example, I have challenged my calculus students with the well-known question concerning cubic functions and tangent lines (Herman 2008; Horwitz 1995; Liu and Blumenthal 2005; Mahoney 2002):

> A cubic polynomial has three distinct real roots. The tangent line drawn to its graph at the point whose x-coordinate is the average of two of these roots intersects the x-axis at the third root (see **fig. 1**). Why does this occur?

This question entices because the result is easy to see yet somewhat peculiar. I usually first ask students to demonstrate this result with a specific cubic polynomial, such as the one shown in **figure 1**. Next, students might try a second verification with a self-constructed cubic. At this point, students may begin to wonder, "Does this always happen?" or even more important, "Why is this occurring?" I have found that students do some of their best mathematics once they realize that they have complete freedom to think. This atmosphere is perfect for stimulating flexible thinking and promoting engagement in meaningful mathematics. When such an opportunity presents itself, I encourage students to study the problem carefully, work in groups, and use technology as they see fit.

This article will share some student-generated ideas that were proposed as starting points in explaining this phenomenon. These responses

TORY NKING ABOUT A NONROUTINE CALCULUS TASK

range from fairly conventional to highly sophisticated and innovative. Rather than presenting the solutions as students might see them, I want to share the inception and formulation of the students' mathematical reasoning. When appropriate, extensions and revisions to students' work are also offered for whole-class discussion. I close with some general reflections on how engagement in such activities may more closely resemble the process of scientific thinking (e.g., see Devlin [2010] for an excellent discussion on this topic).

In my view, proposing such tasks lets students know that we value their internal thinking processes, irrespective of the messiness of the final product. It is no secret that classroom mathematics has done so much of the opposite that students tend to disregard thinking in favor of "quick and proper" solutions (Garofalo 1989; Schoenfeld 1988). In some sense, withholding such exercises from daily classroom activities denies students what is at the heart of what it means to think about and do mathematics (Lockhart 2009; Schoenfeld 1992).

STUDENTS' INITIAL WORK AND THE POTENTIAL FOR DISCUSSION

Brute Force: The Algebra Says So

Probably least elegant is the method in which an arduous algebraic endeavor verifies the statement. That is, students might enter the expression $d \cdot (x - a) \cdot (x - b) \cdot (x - c)$ into a Computer Algebra System (CAS) with a, b, c, and d as constants, $a \neq b \neq c$. Students eventually arrive at a general (and formidable) equation that represents the tangent line at $x = (a + b)/2$:

$$y = -\frac{1}{4}da^2x + \frac{1}{2}dabx - \frac{1}{4}db^2x + \frac{1}{4}da^2c - \frac{1}{2}dabc + \frac{1}{4}db^2c \quad (1)$$

Then it is a straightforward task to let $x = c$ in equation (1) and find that $y = 0$. Although this technique lacks elegance and circumvents the very essence of why such a thing occurs, it does three things for students. First, it connects and communicates the result through two different representations—the algebraic

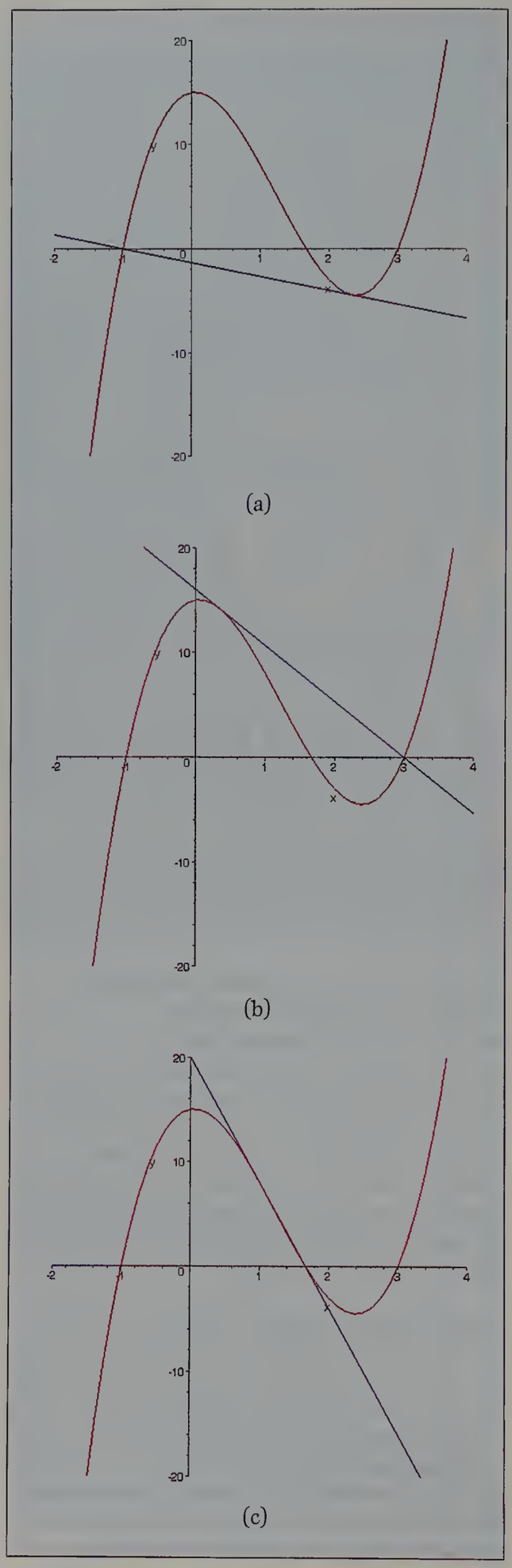


Fig. 1 The graph of $f(x) = 3x^3 - 11x^2 + x + 15$ shows tangent lines drawn at the pairwise averages of two x-intercepts.

and the graphical. Second, it outsources the repetitive routines of a "messy" problem to CAS so that students may focus more intently on the concept. Third, it implicitly tells students that computers are a component of modern mathematical inquiry. Both Horwitz (1995) and Mahoney (2002) support the use of CAS in solving this problem and in fostering the learning of the principles at hand.

The technology—while "solving" the problem in the eyes of students—may well have prevented them from making any significant connections. The nature of the surface complexity of equation (1) makes this difficult to avoid. Because computer technologies often disguise algebraic endeavors, students should be aware of the benefits of expressing equation (1) as

$$y = -\frac{d}{4}(a-b)^2(x-c).$$

It is then apparent that $x = c$ results in $y = 0$. Such an observation can be made with the various options available to CAS users but may be curbed because of CAS's assumed "raw" power. Gently guiding students in these directions may result in a positive and more informed use of technology. Then students may internalize CAS's functionality as a blend of "raw power" and human insight—not a panacea to solving algebraically intractable problems.

Thanks, Newton

Tanya (all names are pseudonyms) found Newton's method for approximating x-intercepts an attractive tool. Newton's method can be stated as follows: Given a function $y = f(x)$, and an initial guess r as a root of $f(x) = 0$, we use the tangent to the curve at $(r, f(r))$ to calculate a subsequent estimate $s = r - f(r)/f'(r)$. Under favorable conditions, $x = s$ is closer than $x = r$ to the sought-after intercept (see **fig. 2**).

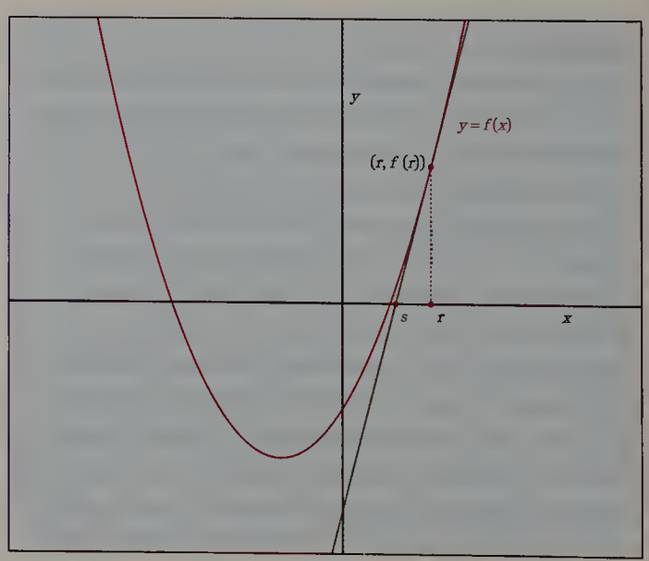


Fig. 2 Newton's method is shown here with initial estimate r and subsequent estimate s.

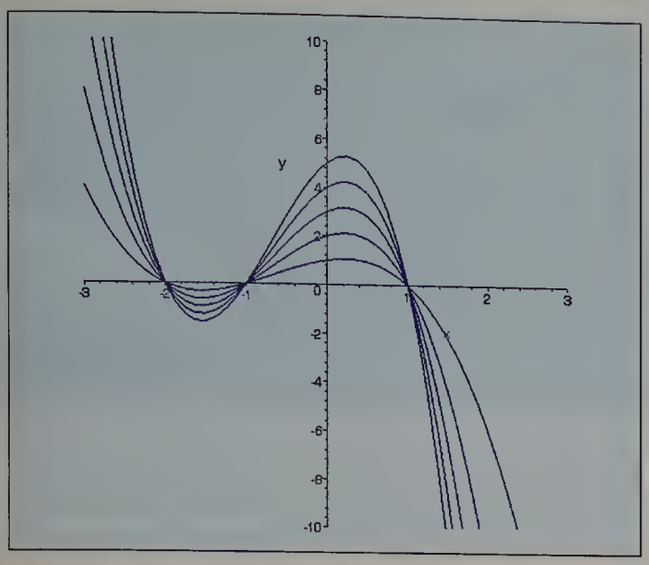


Fig. 3 The graph shows a family of cubics possessing the same x-intercepts.

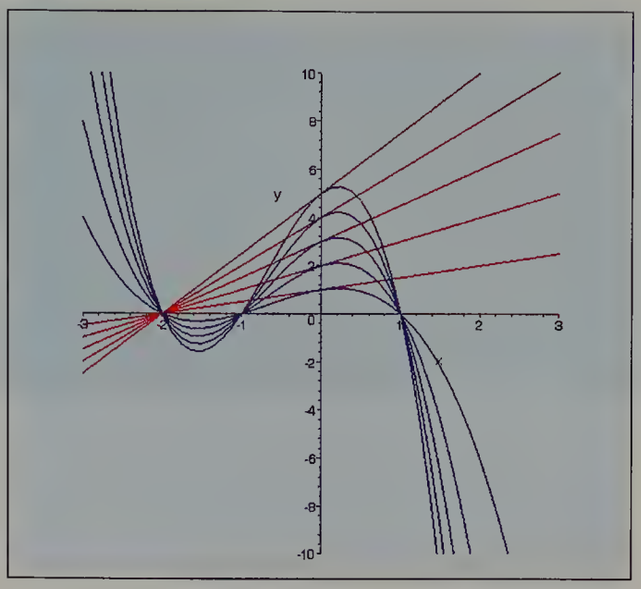


Fig. 4 The family of cubics is pictured with its tangent lines.

"We learned it a while ago, but it's the first thing that came to my head. I mean, this problem is all about zeros [x-intercepts]." Tanya's approach was simple. As an initial guess, she used the average of two zeros. She then performed one iterate of Newton's method, which revealed the other zero. "I knew Newton's method had to have something in the formula that would show me this." Tanya confirmed that her comment was about a "structural clue" in the algorithm $x_{n+1} = x_n - f(x_n)/f'(x_n)$. Sure enough, her work clearly supported this when applied to the function $f(x) = d(x - a) \cdot (x - b)(x - c)$:

$$x_{n+1} = x_n - \frac{d(x_n - a)(x_n - b)(x_n - c)}{d(x_n - b)(x_n - c) + d(x_n - a)(x_n - c) + d(x_n - a)(x_n - b)} \quad (2)$$

Tanya first convinced herself by using two different cubics. She then established this solution on more general grounds by generating equation (2). Tanya remarked that with initial guess $x_0 = (a + b)/2$, Newton's method returned $x_1 = c$. Subsequent iteration also returned this value. After the solution was presented to her classmates, most of them needed a moment to grasp it. But shortly after, a good number asked sheepishly, "Why didn't I think of that?"

Tanya's insight can be used to provide classmates with an even more general approach that connects nicely with quadratics. Consider the cubic $p(x) = x(x^2 + bx + c)$. No generality is sacrificed here, because we may scale the leading coefficient to 1 and horizontally translate the cubic so that one of its intercepts occurs at the origin. At this point, students may need a brief reminder that the midpoint of the intercepts of the graph of $y = x^2 + bx + c$ occurs when $x = -b/(2 \cdot 1) = -b/2$. Applying Newton's method to p with initial guess r yields

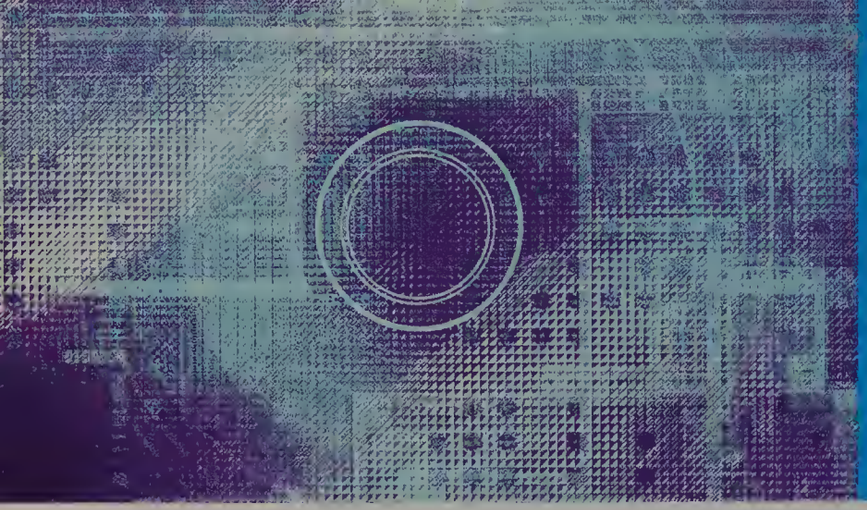
$$s = r - \frac{p(r)}{p'(r)} = r - \frac{r(r^2 + br + c)}{3r^2 + 2br + c} = \frac{2r^3 + br^2}{3r^2 + 2br + c} = \frac{r^2(2r + b)}{3r^2 + 2br + c}. \quad (3)$$

The interesting revelation here is that the subsequent estimate s is zero provided that the initial guess is $r = -b/2$. So, indeed, if we choose the average of the two nonzero roots, the tangent line drawn at this location passes through the origin. What is appealing in this approach is that the initial guess r need not be prescribed a priori. Once equation (3) is established, we can see that $r^2(2r + b) = 0$ when $r = -b/2$.

A Critical but Qualitative Approach

Shelly and Johanna collaborated on this problem, finding a graphical approach most appealing. To examine this phenomenon, they sketched a family of cubics having intercepts $x = -2$, -1, and 1 (see **fig. 3**). They chose this family for the simple reason that averaging –1 and +1 results in $x = 0$, a value likely to maximize computational efficiency.

At this stage, Shelly and Johanna carefully examined how the curves "bent" at precisely where they intersected the y-axis (this bending phenomenon shed light on the tangent line's steepness at $x = 0$). What they found was a steeper tangent as the y-intercept increased. This discovery, in turn, made credible the idea that the tangent line at $x = 0$ for the entire family had the identical striking point on the x-axis—the other intercept (see **fig. 4**). Although many would judge this approach as lacking rigor, we need glance only as far as shooting methods in numerical analysis or other root-finding algorithms to be reminded that this idea could be the seed of something far more complex.



I sometimes offer tasks as optional take-home assignments, dangling a carrot such as, "Everyone in the room can do this. You just need to think about it."

On the downside, neither Shelly nor Johanna appeared aware of the specificity of the chosen polynomial. That is, both believed that demonstrating this graphical relationship for the polynomial family $p(x) = k(x + 2)(x + 1)(x - 1)$ was sufficient to "prove" this phenomenon.

Two interesting discussions grow directly from this conviction. On the local level, all the graphs displayed are vertical dilations of the parent function $p(x) = (x + 2)(x + 1)(x - 1)$. Hence, the tangents displayed in the graph always intersect the curves at the same location on the x-axis. Here we could ask students why they might anticipate such regularity (i.e., preservation of tangency for the family). On the generative level, because students were not aware of the insufficiency of using a single example, this was an opportunity to discuss some traditions in mathematics with respect to mathematical proof.

Shelly and Johanna's initial investigation was not a lost cause. With further guidance, they could generalize their original choice for a polynomial to $p(x) = (x - 1)(x - a)(x + a) = x^3 - x^2 - a^2x + a^2$. That is, they could choose two of the intercepts to straddle the origin and scale the polynomial so that its remaining root is $x = 1$. Here, we find that $p(0) = a^2$ and $p'(0) = -a^2$, resulting in the tangent line $y - a^2 = -a^2(x - 0)$, so the conclusion follows immediately. Shelly and Johanna's dabbling with a specific example can open the door to a fuller understanding of what the task entails. With some teacher assistance, students may then be able to better evaluate the generative power of their chosen method.

Limits and Epsilons

Jalen chose arguably the most interesting pathway of them all. He used the form $f(x) = d(x - a)(x - b) \cdot (x - (b + \varepsilon))$ for a small positive number ε. The goal here was to maintain the assumptions in the question (three distinct zeros) but to examine the tangent line behavior as two distinct zeros began to coincide. Jalen went through the process of finding the equation for the tangent line (in general) to $y = f(x)$ and then examined the line's behavior for $\varepsilon \to 0$. Believing that he had an enormous algebraic task before him, he began with a concrete example: $p(x) = (x - 2)(x + 3)(x + (3 + \varepsilon))$. In contrast with some other solution methods, Jalen found the "three distinct x-intercepts" part of the statement to be most significant. He remarked that if he could examine this boundary situation (with $\varepsilon = 0$), this investigation might provide a means for finding a solution.

To build a solution, Jalen averaged the two zeros closest in magnitude, $x = -3$ and $x = -3 - \varepsilon$, getting $x = -(6 + \varepsilon)/2$. He then used a symbolic utility to find $p'(x)$ and the equation of the tangent line to the graph of p at $x = -(6 + \varepsilon)/2$, resulting in the tangent line equation $y = -(\varepsilon^2/4)(x - 2)$. At this point in his thinking, Jalen consulted me, expressing his embarrassment in "using a limit." I recommended that he examine cases for which $\varepsilon \to 0$. He immediately replied that doing so would result in "tangent lines being more and more horizontal." Jalen shared the graphs shown in **figure 5**, which supported his approach.

Jalen had initial doubts, in the case of a double root, that the original phenomenon would be observed and was surprised to find that the limiting case held true. Jalen's situation, which is analogous to Shelly and Johanna's, can be captured by a transformation such as $r(x) = x^2(x - 2)$. Because of the double root, there are only two cases to examine: one with a tangent at $x = 0$ and the other with a tangent at $x = 1$. Jalen's analysis provides evidence of the former case, whereas computing $r(1)$ and $r'(1)$ establishes that the tangent to the graph at $x = 1$ passes through the "other" zero at $x = 0$.

Jalen did not solve the original problem as stated, but his struggle was a productive one. Through experimentation, he demonstrated the following result: As two roots coincide, the limiting position of the x-intercept of the tangent is the third root. As his teacher, I have little doubt that "playing around" in a meaningful way was mathematically productive and resulted in augmenting his understanding of the original task. Like professional mathematicians, Jalen felt the need to convince himself of the veracity of the statement before laying out a blueprint to prove it, even if he never reached this stage.

For a different student solution that also uses limits, see the online extra at **www.nctm.org/mt033**.

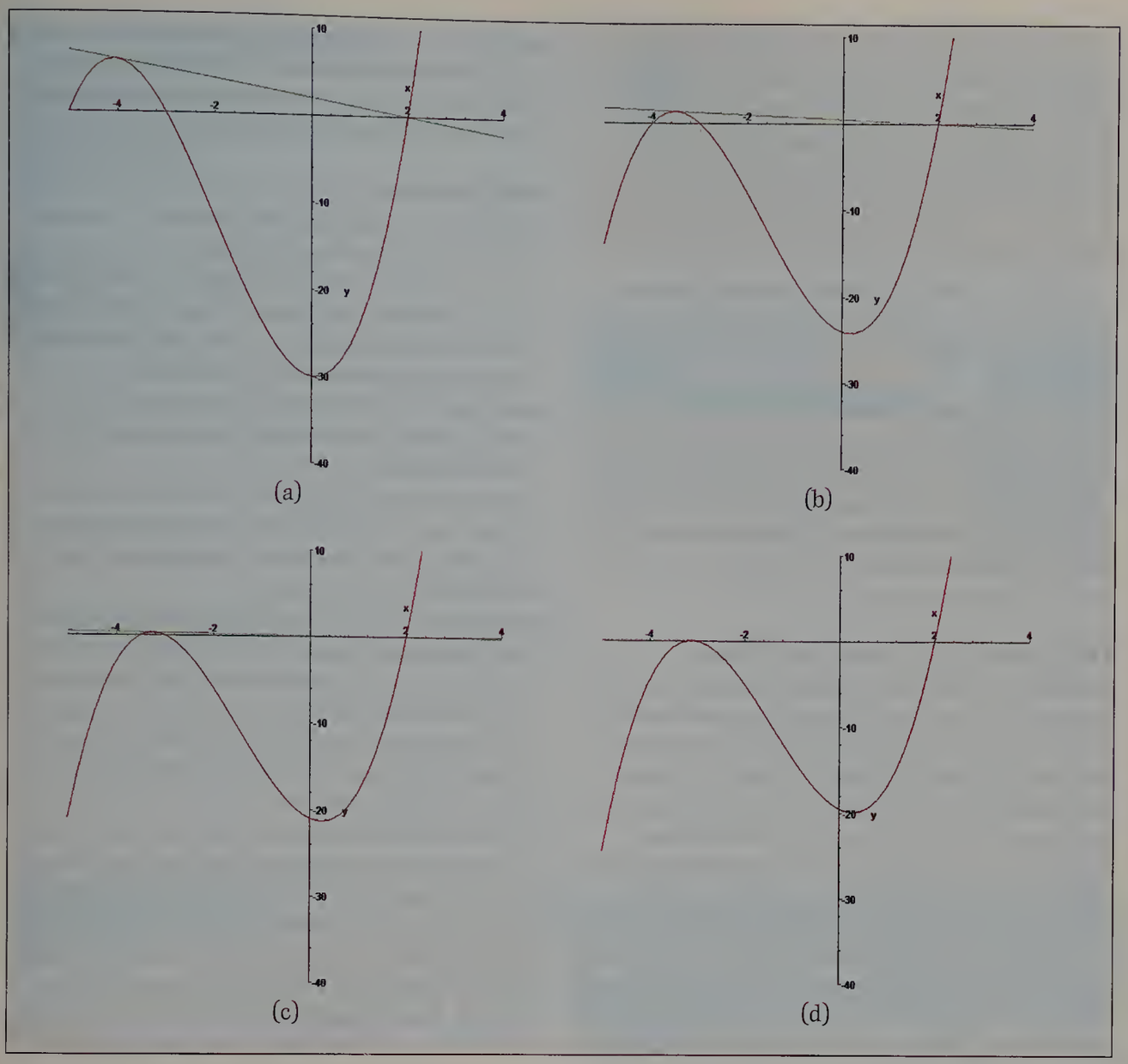


Fig. 5 The graph of $p(x) = (x - 2)(x + 3)(x + (3 + \varepsilon))$ is shown with its tangent line $y = -(\varepsilon^2/4)(x - 2)$ for $\varepsilon = 2$ (**a**), 1 (**b**), 0.5 (**c**), and 0.25 (**d**).

DISCUSSION

The variety of the students' solutions speaks for itself. Exercises such as these may not teach content per se but may leave students with a renewed sense and awareness of the nature of a mathematician's work. Trinter and Garofalo (2011) have recently reported that nonroutine problems in algebra may alter students' impressions of mathematics and allow for a keener sense of understanding and more productive lines of thinking. Perhaps the greatest benefit to teachers is seeing the application of important mathematical ideas outside the narrow context in which they are often introduced: Tanya used a root-finding algorithm, Shelly and Johanna examined a family of functions, and Jalen used a limit. None of the students was told to use these tools; instead, they found them useful on their own.

Finding time to include such tasks in an already-crowded curriculum may seem challenging. However, I sometimes offer similar tasks as optional take-home assignments, dangling a carrot such as, "Everyone in the room can do this. You just need to think about it." Giving students about a week to complete the assignment sends a few tacit messages, including these:

- Take your time.
- Experiment and reflect on your findings.
- Ask me (and your classmates!) questions.
- Share ideas.

Sharing solutions does not consume much classroom time once students have attempted the problem and expressed an intellectual investment in its solution. As a teacher, I have learned that problems that appear somewhat nonroutine but contain elements of familiarity may provide all the incentive a student needs to try. Students may surprise their teachers with the originality they bring to otherwise ordinary-looking problem situations.

Finally, although this problem encourages ingenuity and creativity, it could be argued that students were hand-delivered the result—that is, they were asked to prove or explain a predefined phenomenon. As an alternative, the exploration factor can be heightened by stating the problem in reverse:

> Consider a cubic function whose graph has three distinct x-intercepts. Draw a tangent line from one of the x-intercepts of the graph. What can be said about its point of tangency on the graph?

This reformulation invites students to examine an outcome most likely expedited by looking at several cases. From this, students may extract the phenomenon as a relevant characteristic of *all* cubic functions (as opposed to being told). Incorporating this extra window of opportunity also invites informed use of dynamic software.

THE VALUE OF A GOOD QUESTION

Each time I assign this exercise, I am reminded of all the good that can come from asking a question that is interesting, challenging, and open-ended. Most important, by providing no road map to a solution, the problem invites students to think in nonprescriptive ways. The array of starting points discussed here has convinced me that a lot of interesting mathematics is sitting dormant in students' minds. We will never see this mathematics if we don't ask.

Too often, mathematicians highlight the beautiful results of mathematics but suppress the arduous journey in getting there. As Tall (1991) once said, "Many mathematicians have learned to present their best face in public, showing their ideas in polished form and concealing the toil and false turnings that littered their growth" (p. 17). This article reports on the opposite idea. Students were able to construct interesting starting points and potential pathways to a solution. The students who never solved the problem (Shelly and Johanna as well as Jalen) were still proud of their initial work. Some felt that their ideas were generalizable. For other students, the experience of being stuck was healthy and surprisingly satisfying.

Problems like the one discussed here are still rare in the modern curriculum, an unfortunate situation given the variety of interesting and nonroutine tasks that can be found in the literature. These tasks are not limited to calculus but include problems from algebra, geometry, number theory, and the like.

For me, a good problem offers an experience that mirrors the history of mathematics, the evolution of scientific thinking, and the very act of what it means to *do* mathematics (Devlin 2010). The clashing of success and failure amid the struggle to explain allows students to test their mathematical arsenal and catch a glimpse into the true nature of mathematics. Offering similar exercises in calculus and more widely across the curriculum promises a wealth of opportunities with respect to students' emergent thinking and mathematical growth.

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SOURCES FOR NONROUTINE PROBLEMS

Many standard undergraduate calculus texts contain sections titled Problems Plus, Problem Solving, Additional Exercises, or something similar. These sections contain problems that require insight and ingenuity on the students' part.

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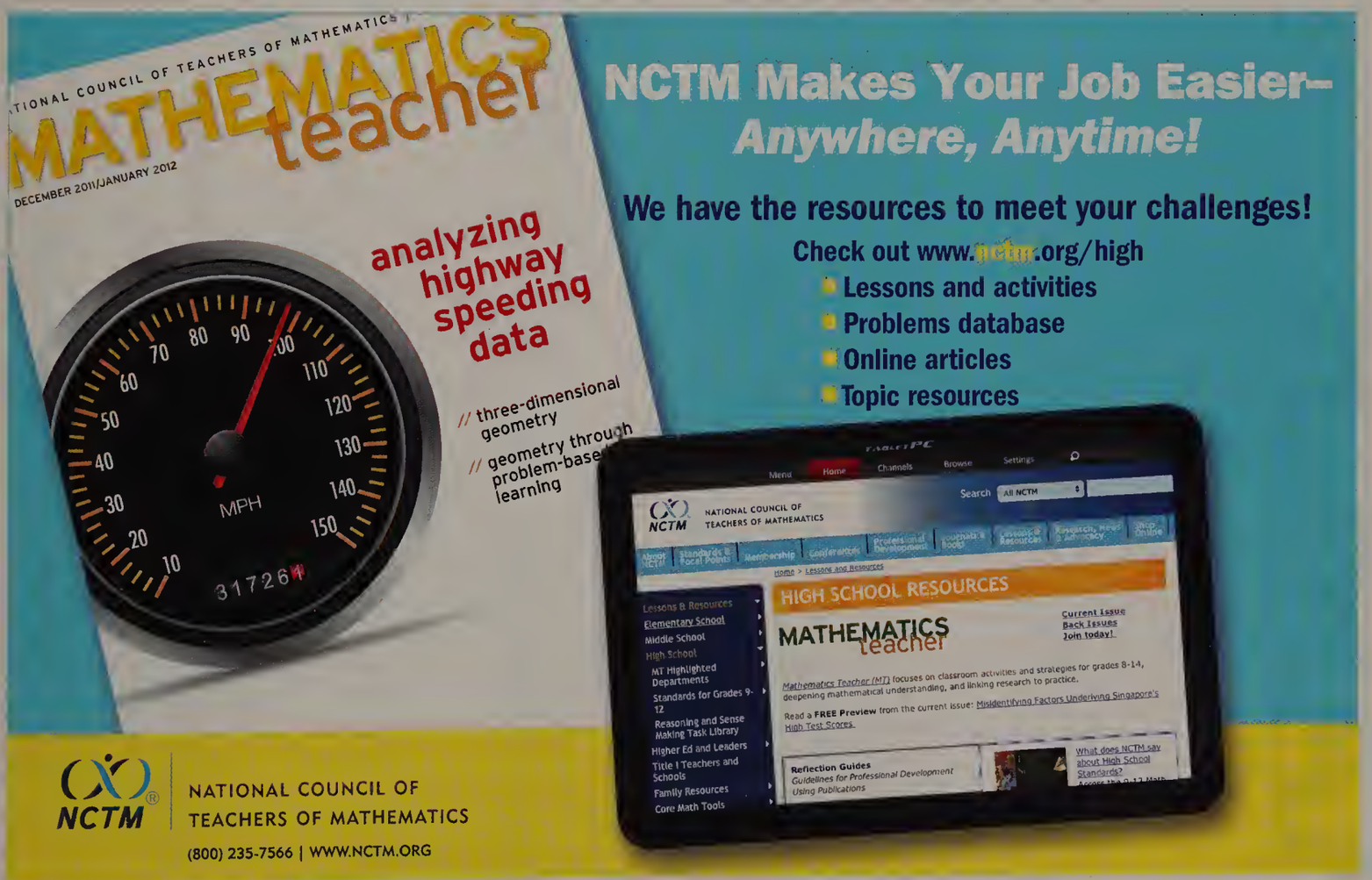
more4U

For an additional student solution that also uses limits, download one of the free apps for your smartphone and then scan this tag to access **www.nctm.org/mt033**.



MARCH

NCTM® NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

The year 2013 contains four digits whose values are consecutive integers. How many years must we wait for this event to occur again?

1

How many years ago did we last have a year with digits whose values are consecutive integers?

2

Does the number 2013 belong in the following sequence?

2, 5, 8, 11, 14, ...

3

Find the smallest positive integer n such that 2013 is a term in the following sequence:

$$n, n+7, n+14, n+21, \cdots$$

4

Find the sum of the prime factors (with repeats if necessary) of 2013. Then find the sum of the prime factors, or *factor sum*, of your result. Continue in this fashion until you reach a fixed, unchanging value. What is that number?

5

If we add the two factor sums in the problem for March 5, we get $75 + 13 = 88 \neq 2013$, the initial integer. Can you find three composite numbers, one between 70 and 80 and two less than 40, such that the sum of the factor sums equals the initial number?

6

Given nine nickels and four pennies, find how many different sums of money can be formed using one or more coins.

7

The product of a two-digit number and the same number with its digits reversed is 3154. Find the sum of the two numbers.

8

If a, b, c, and d represent the digits of the number 2013, find all possible values of the expression

$$\left(\left(a^b\right)^c\right)^d.$$

9

The numbers from 1 to 100, inclusive, are written out in base 10. Compute the sum of all the digits in these numbers.

10

The numbers from 1 to 100, inclusive, are written out in base 8. Compute the base-10 sum of all the digits in these numbers.

11

The old 30-question AHSME (American High School Mathematics Exam) was graded as follows: Students were awarded 5 points for a correct answer, 0 points for a wrong answer, and 2 points if left blank. The maximum score was 150, and the minimum score was 0. Find all integer scores between these values that are not possible.

12

Given the sequence of alternating terms

$$A = 5, 9, 5, 9, 5, 9, \ldots,$$

find an expression for a_n, the nth term in the sequence, in terms of n.

13

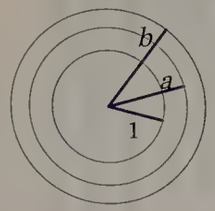
In how many ways can you use quarters and dimes to create an amount equal to \$20?

14

A *palindromic number* is a number that reads the same forward and backward. Some examples include 121, 464, and 2,345,432. What integer greater than 1 divides every four-digit palindrome?

15

A circle of radius 1 is concentric with circles of radii a and b with $b > a > 1$. The area of the smallest circle equals the area of the ring (the *annulus*) with outer radius a and inner radius 1 and also equals the area of the ring with outer radius b and inner radius a. Find a and b.



16

Take a two-digit number, represented by ab, such that neither a nor b is zero and $a \neq b$. Reverse the digits and add the result to the original number. Divide this sum by the sum $a + b$. What is the quotient? Prove that this holds true for all such two-digit numbers.

17

The sum of every possible arrangement of three-digit numbers each consisting of the distinct, nonzero digits a, b, and c is divided by the sum of the three digits. Determine the quotient.

18

The numerical value of the volume of a cylinder is twice the numerical value of its surface area. If the radius and the height are both integers, determine the smallest possible volume.

19

Annette is a trained ant whose trick consists of walking along graph paper. She starts at the origin, walks 16 units right, and then walks 2 units down to reach (16, –2). She then walks 8 units left and 2 units down to reach (8, –4). She alternates right and left, cutting the horizontal distance in half each time and always walking down 2 units. Which lattice point will she be closest to after 10 complete moves?

20

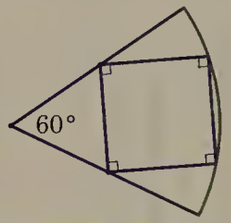
Given the operation $\triangle$ defined as

$$x \triangle y = x^2 - xy + y^2,$$

find the values of x and y if $x \triangle y = 768$ when $x = 2y$.

21

A square is inscribed in a sector of a circle such that two vertices lie on the arc of the sector and two vertices lie on the radial sides. If the central angle of the sector is 60°, find the ratio of the area of the square to the area of the sector.



22

Find the quotient when $(a^4 - b^4)$ is divided by $(a - b)$.

23

Circle O has radius 6. Congruent circles P and Q are internally tangent to circle O and externally tangent to each other. Circle S is internally tangent to circle O and externally tangent to circles P and Q. Find the radius of circle S.

24

Q and R with coordinates (2, 2) and (10, 8), respectively, are the endpoints of the hypotenuse of right triangle PQR. What is the maximum area enclosed by $\triangle PQR$?

25

Luka and Janie are playing a coin toss game. If the coin lands heads up, Luka earns a point; otherwise, Janie earns a point. The first player to reach 20 points wins the game. If 19 of the first 37 tosses have been heads, what is the probability that Janie wins the game?

26

$MRCH$ is a square in the coordinate plane such that M is at the origin and C, the opposite vertex, has coordinates (a, a), where $a > 0$. Point X is the midpoint of $\overline{MR}$, and Y is the midpoint of $\overline{MH}$. Segments CX and CY are drawn. Diagonal RH intersects $\overline{CX}$ and $\overline{CY}$ at P and Q, respectively. Find the coordinates of P and Q in terms of a.

27

In the array of odd integers below, in which row will 2013 appear?

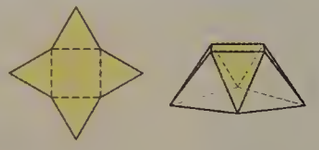
			1			
		3	5	7		
	9	1	1	1	1	
		1	3	5	7	
1	2	2	2	2	2	3
9	1	3	5	7	9	1
			⋮			

28

On an 8-in. square of paper, construct the four segments that connect the midpoints of consecutive sides. Crease the paper on those lines so that the four triangular "flaps" are perpendicular to a square base. The four vertices of the original square are now the vertices of a smaller square, in a plane parallel to the original square. Find the length of a side of the smaller square.

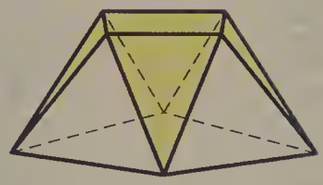
29

Fold a square of paper as described in the problem for March 29. If a figure similar to the one shown on the left is cut out and folded on the dashed segments, the two folded figures will fit together perfectly. They form a three-dimensional figure called a *prismatoid*. Find the total surface area of the prismatoid.



30

Determine the volume of the prismatoid that you found in the problem for March 30.



31

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SOLUTIONS to calendar

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The Editorial Panel of *Mathematics Teacher* is considering sets of problems submitted by individuals, classes of prospective teachers, and mathematics clubs for publication in the monthly Calendar. Send problems to the Calendar editors. Remember to include a complete solution for each problem submitted.

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1. 18. The next year consisting of digits that are consecutive integers is 2031, so we must wait 18 years.

2. 581. The last year that consisted of four consecutive integers was 1432, 581 years ago.

3. No. The terms in the sequence increase by 3, yet none of the numbers is divisible by 3. The number 2013 is divisible by 3 since the sum of its digits is divisible by 3, so it is not in the sequence.

4. 4. The common difference between terms is 7. Dividing 2013 by 7 leaves a remainder of 4. Thus, $4 + 287(7) = 2013$, and $n = 4$.

5. 13. The prime factors of 2013 are 3, 11, and 61; thus, $3 + 11 + 61 = 75$. The prime factors of 75 are 3, 5, and 5; thus, $3 + 5 + 5 = 13$. Since 13 is prime, the sum of the prime factors remains constant at 13.

6. 74, 38, 20. Since 71, 73, and 79 are prime, we must consider six composite numbers between 70 and 80. If we test 72, we have $72 = 2^3 + 3^2$ and $2 + 2 + 2 + 3 + 3 = 12$. Then $12 = 2^2 + 3$ and $2 + 2 + 3 = 7$. The sum of the factor sums is $12 + 7 = 19 \neq 72$. If we test 74, we find that we need five iterations of the procedure to reach a prime:

$74 = 2 \cdot 37$ and $2 + 37 = 39$
$39 = 3 \cdot 13$ and $3 + 13 = 16$
$16 = 2^4$ and $2 + 2 + 2 + 2 = 8$
$8 = 2^3$ and $2 + 2 + 2 = 6$
$6 = 2 \cdot 3$ and $2 + 3 = 5$

The sum of the factor sums is $39 + 16 + 8 + 6 + 5 = 74$, the initial value. The two integers less than 40 with this property are 38 and 20. The factor sums for 38 are 21, 10, and 7; those for 20 are 9, 6, and 5. These three integers are the only integers less than 1,000,000 that have this property.

7. 49. Every sum can be formed from 1 cent to 49 cents, so there are 49 different sums.

8. 121. Factor 3154 into $2 \cdot 19 \cdot 83$ and observe that $2 \cdot 19 = 38$, which is the reverse of 83. The sum of 38 and 83 is 121.

Alternate solution: Let one of the numbers be $10t + u$; then the number with the digits reversed will be $10u + t$. Therefore, $10t^2 + 10u^2 + 101tu = 3154$. Since the first two terms have zeros in the units place, it is the third term that produces the 4 in the units place. So $tu = 4$, 14, or 24. It is easy to see that if $tu = 4$, the number pairs would be either 22 and 22 or 14 and 41. But $22^2 = 484$, and $14(41) = 574$. If $tu = 14$, the number pairs would be 27 and 72, with a product of 1944. Therefore, $tu = 24$, and the number pairs are either 46 and 64 or 38 and 83. Since $46(64) = 2944$, the two numbers must be 38 and 83, and their product is 3154.

9. 0 and 1. We can rewrite

$$\left(\left(a^b\right)^c\right)^d$$

as a^{bcd}. If $a = 0$, then $bcd = 6 \neq 0$, and the value of the expression is 0. If $a \neq 0$, then $bcd = 0$, and the value of the expression is 1.

10. 901. The sum of the digits from 1 to 9 is 45. The sum of the digits from 10 to 19 is $10(1) + 45 = 55$. Likewise, the sum of the digits from 20 to 29 is $10(2) + 45 = 65$. We continue in this fashion until we find the sum of the digits from 90 to 99: $10(9) + 45 = 135$. Add the numbers $45 + 55 + 65 + \cdots + 135 = 900$, and then add 1 from the number 100 to get 901.

11. 675. Arrange the numbers in rows as shown. The units digits in each row, excluding the last row, add to 28. The tens digits in each column for the first eight rows also sum to 28. Therefore, the sum of the digits in the first eight rows is $2 \cdot 8 \cdot 28 = 448_{10}$. Using the same ideas for the next four rows, we have $4 \cdot 28 + 8 \cdot 6 + 8 \cdot 4 = 192$. The sum of

the digits in the last row is 5 + 6 + 7 + 8 + 9 = 35. Therefore, the sum of all the digits is 448 + 192 + 35 = 675_{10}.

0	1	2	3	4	5	6	7
10	11	12	13	14	15	16	17
20	21	22	23	24	25	26	27
30	31	32	33	34	35	36	37
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
70	71	72	73	74	75	76	77
100	101	102	103	104	105	106	107
110	111	112	113	114	115	116	117
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
140	141	142	143	144			

12. 1, 3, 143, 146, 148, and 149. If one answer is left blank and the rest are wrong, the score is 2. Every even score up to 60 can be reached by leaving one additional answer blank. If one answer is correct and the rest are wrong, the score is 5. Every odd score from 5 to 63 can be reached by leaving one additional answer blank. Every even score from 62 to 144 can be reached through a combination of an even number of correct answers—at least 12 correct—and up to 18 left blank. Why is 144 the maximum? If 28 answers are correct, then only 2 questions can be left blank. Odd scores from 65 to 141 and the scores 145 and 147 can be reached with a combination of an odd number of correct answers—between 13 and 29—and at most 17 left blank. Why isn't a score of 143 possible? If 27 answers are correct, then at most 3 can be left blank. The same reasoning applies to the impossible score of 149. Note that there is no way to obtain a score of 1 or 3.

13. $a_n = 7 + 2(-1)^n$ or $a_n = 7 + 2\cos(180n)°$. The number 7 is midway between $a_1 = 5$ and $a_2 = 9$, so we can either add 2 or subtract 2 from 7. The idea of adding or subtracting can be controlled by raising –1 to odd or even powers. Therefore, one possible answer is $a_n = 7 + 2(-1)^n$. Another possible solution is $a_n = 7 + 2\cos(180n)°$.

14. 41. Consider the simpler task of finding the number of ways to use quarters and dimes to create $1.00. We can use 4 quarters, 2 quarters, or 0 quarters. Only an even number of quarters will work. To make $20, we need 80 or 78 or 76, …, or 2 or 0 quarters. There are 41 even integers from 0 to 80, inclusive.

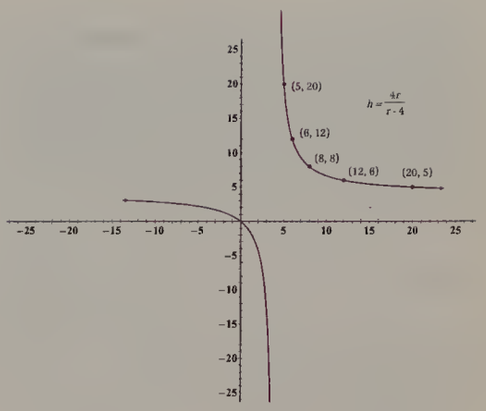
15. 11. Create a few four-digit palindromes, factor them, and look for a common factor. For example: 1551 = 3 • 11 • 47, and 2002 = 2 • 7 • 11 • 13. Only 11 is a factor of both palindromes. Why must 11 be a factor? A test for divisibility by 11 begins by adding every other digit and subtracting the sum of the remaining digits. If the difference is a multiple of 11, then so is the original number. For example, 23,456 is not a multiple of 11 since (2 + 4 + 6) – (3 + 5) = 4, which is not a multiple of 11. However, 234,564 is a multiple of 11 because (2 + 4 + 6) – (3 + 5 + 4) = 0, which is a multiple of 11. Since all four-digit palindromes are of the form *abba* and (*a* + *b*) – (*a* + *b*) = 0, every such number will contain a factor of 11.

16. $a = \sqrt{2}$, and $b = \sqrt{3}$. Since the radius of the inner circle is 1, its area is π. Therefore, the area of the middle ring is $\pi a^2 - \pi = \pi \rightarrow a^2 = 2 \rightarrow a = \sqrt{2}$. Similarly, the outer ring has area $\pi b^2 - 2\pi = \pi \rightarrow b^2 = 3 \rightarrow b = \sqrt{3}$.

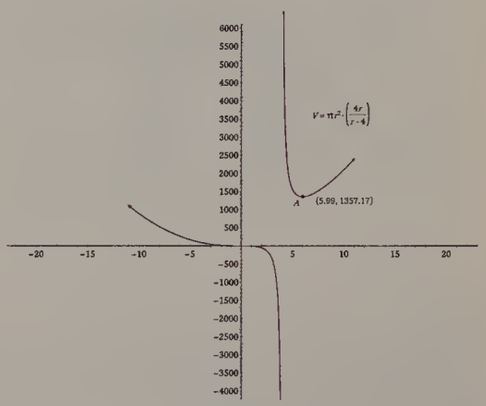
17. 11. Let 10*a* + *b* = the value of the original two-digit number. Reversing the digits gives us 10*b* + *a*. Their sum is 11*a* + 11*b*, which, when divided by (*a* + *b*), has a quotient of 11.

18. 222. There are six permutations of the letters *abc*, each producing a different number. Each letter appears in one of the three positions exactly twice. The number *abc* can also be written as 100*a* + 10*b* + *c*, so the sum of all six numbers is 2 • 100(*a* + *b* + *c*) + 2 • 10(*a* + *b* + *c*) + 2(*a* + *b* + *c*) = 222 • (*a* + *b* + *c*). When this sum is divided by (*a* + *b* + *c*), the quotient is 222.

19. $500\pi \approx 1571$ units^3. Since the volume of a cylinder is $V = \pi r^2 h$ and its surface area is $SA = 2\pi rh + 2\pi r^2$, we have $\pi r^2 h = 4\pi rh + 4\pi r^2$; thus, $h = 4r/(r - 4)$. Since volume is directly proportional to the square of the radius for a given height, *r* needs to be as small as possible to minimize volume. Note also that the expression $h = 4r/(r - 4)$ implies that $r > 4$. Let $r = 5$, thus implying that $h = 20$ and $V = 500\pi \approx 1571$ units^3. The figure (top of next column) shows that there are exactly five integral solutions to the equation $\pi r^2 h = 4\pi rh + 4\pi r^2$.



If we eliminate the restriction that the radius and the height must be integers, we can use graphing software to find a smaller volume, approximately 1357.17 units^3, when *r* is slightly less than 6. See the figure below.



20. (11, –20). The ant's location in the *x*-direction after 10 moves is given by the sum $S = 16 - 16/2 + 16/4 - 16/8 + \cdots - 16/512$. This is a geometric series with $a_1 = 16$, $r = -1/2$, and $n = 10$. Find the sum directly or use the formula for the sum of *n* terms of a geometric series:

$$S = a_1\left(\frac{1-r^n}{1-r}\right) = 16\left(\frac{1-(-0.5)^{10}}{1-(-0.5)}\right)$$

$$= \frac{32}{3}\left(1-\frac{1}{1024}\right) = \frac{32}{3}\cdot\frac{1023}{1024} = \frac{341}{32},$$

which rounds to 11. Since Annette always moves down 2 units, her *y*-coordinate is exactly –20.

21. (32, 16) and (–32, –16). We have $x \triangle y = 768 = x^2 - xy + y^2$. Substitute 2*y* for *x* to obtain the following: $(2y)^2 - (2y)y + y^2 = 768 \rightarrow 4y^2 - 2y^2 + y^2 = 768 \rightarrow y^2 = 256 \rightarrow y \pm 16$. So the required solutions (*x*, *y*) are (32, 16) and (–32, –16).

22. Approximately 0.5117. Let *s* equal

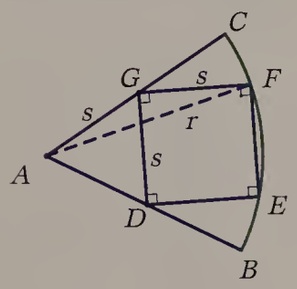
a side of the square, and let r equal the radius of the sector. Use the symmetry of the figure and the fact that $m\angle CAB = 60°$ to show that $\triangle GAD$ is equilateral. Therefore, $AG = s$. We know that $m\angle AGF = 150°$, so, by the law of cosines,

$$r^2 = 2s^2 - 2s^2\cos(150°) = s^2\left(2 + 2 \cdot \frac{\sqrt{3}}{2}\right).$$

We want the ratio of the areas:

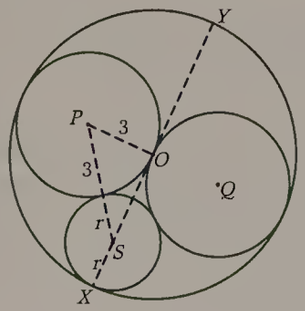
$$\frac{A_{square}}{A_{sector}} = \frac{s^2}{\frac{\pi r^2}{6}} = \frac{6s^2}{\pi r^2} = \frac{6s^2}{\pi s^2\left(2+\sqrt{3}\right)}$$

$$= \frac{6}{\pi\left(2+\sqrt{3}\right)} \approx 0.5117$$



23. $(a^3 + a^2b + ab^2 + b^3) = (a + b) \cdot (a^2 + b^2)$. Factor the difference: $(a^4 - b^4) = (a^2 - b^2)(a^2 + b^2) = (a - b)(a + b) \cdot (a^2 + b^2)$. Observe that $(a - b)$ appears as a factor. Multiply as follows: $(a + b) \cdot (a^2 + b^2) = (a^3 + a^2b + ab^2 + b^3)$.

24. 2. Since the radius of circle O is 6, the radius of circle P is 3. The common internal tangent to circles P and Q (segment XY in the figure) is perpendicular to $\overline{PO}$ and, by symmetry, contains point S. Let r be the radius of circle S. By the Pythagorean theorem, $PO^2 + OS^2 = PS^2 \rightarrow 3^2 + OS^2 = (3 + r)^2 \rightarrow OS^2 = 6r + r^2$. But $OS + r = 6 \rightarrow OS^2 = 36 - 12r + r^2$. Set these two expressions for OS^2 equal to each other: $6r + r^2 = 36 - 12r + r^2 \rightarrow r = 2$.



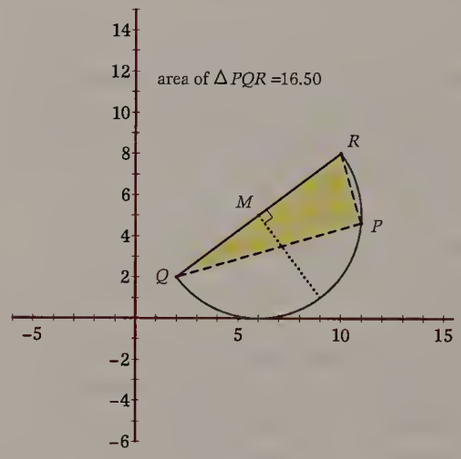
25. 25. For all right triangles with a given hypotenuse, the right triangle with the greatest area has equal legs—a right triangle with sides in ratio $1:1:\sqrt{2}$. Use the distance formula to find the length of hypotenuse QR:

$$\sqrt{(10-2)^2 + (8-2)^2} = 10$$

Each leg has length $10/\sqrt{2}$, so the maximum area enclosed by $\triangle PQR$ is

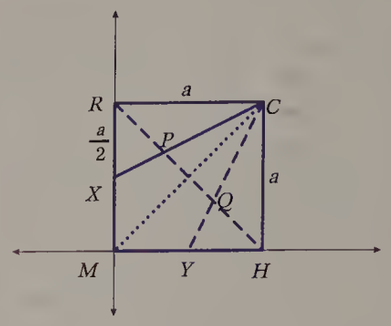
$$\frac{\left(\frac{10}{\sqrt{2}}\right)^2}{2} = 25.$$

Alternate solution: Plot Q and R and construct $\overline{QR}$. Assume temporarily that P has coordinates (10, 2). Then $\triangle PQR$ is a right triangle with $QP = 8$ and $RP = 6$. Familiarity with Pythagorean triples gives us $QR = 10$ by inspection. The midpoint of $\overline{QR}$ is M (6, 5). All points P such that $\triangle PQR$ is a right triangle lie on the circle containing Q and R with radius = $QR/2$ = 5 and center M. If $\overline{PM} \perp \overline{QR}$, then $PR = PQ$, and $\triangle PQR$ has the maximum area: $(1/2)PM \cdot QR = (1/2)(5)(10) = 25$.



26. 1/4. If 19 of the first 37 tosses were heads, then Luka has 19 points, and Janie has 18 points. The next two tosses must be tails for Janie to win. Since coin tosses are independent events, we have $P(\text{tails, tails}) = 1/2 \cdot 1/2 = 1/4$.

27. $P(a/3, 2a/3)$, $Q(2a/3, a/3)$. Sketch the figure. Since X is the midpoint of $\overline{MR}$, the distance from X to R is $a/2$. The slope of $\overleftrightarrow{CX}$ is $(a/2)/a = 1/2$; the equation for $\overleftrightarrow{CX}$ is $y = x/2 + a/2$. Diagonal RH has the equation $y = -x + a$. $\overline{RH}$ and $\overleftrightarrow{CX}$ intersect at P; therefore, $x/2 + a/2 = -x + a \rightarrow 3x/2 = a/2 \rightarrow x = a/3$. Substitute to find the y-coordinate: $y = -x + a \rightarrow y = -(a/3) + a \rightarrow y = 2a/3$. So the coordinates of P are $(a/3, 2a/3)$. Point Q is the reflection of P in the line $y = x$; the coordinates of Q must be $(2a/3, a/3)$.

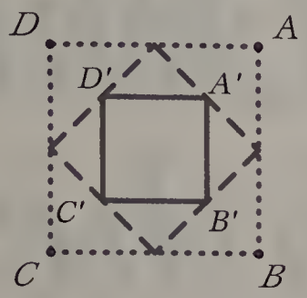


28. Row 32. We would prefer to solve this problem without writing out all the odd integers from 1 to 2013. A first step in decreasing the workload might be to find a pattern to generate the final integer in each row: 1, 7, 17, 31, …. Observe that we add 6, then 10, then 14. We predict that we will add 18 to 31 to obtain the final digit in the next row—namely, 49—and we confirm that prediction. How can we avoid adding integers that increase by 4 until we reach the row containing 2013? The method of finite differences tells us that a formula for the last integer in each row will be quadratic. So we examine the integer pairs of the form (row number squared, last integer in row): (1, 1), (4, 7), (9, 17), (16, 31). If we square the row number, multiply by 2, and subtract 1, we obtain the final integer in that row. So we guess row 25: $2(25)^2 - 1 = 1249$, which is too small. We then guess row 30: $2(30)^2 = 1799$. We then guess row 32: $2(32)^2 = 2047$. Row 32 contains 63 odd integers, so we know that 2013 will be found in this row.

Alternate solution: Observe that there is 1 integer in the first row, 4 integers in the first two rows, 9 integers in the first three rows, and 16 integers in the first four rows. Therefore, in general, there are n^2 integers in the first n rows. Since 2013 is the 1007th odd integer, we wish to find the integer n such that $(n - 1)^2 \le 1007 \le n^2$. We have $31^2 = 961 \le 1007 \le 1024 = 32^2$.

29. 4 in. Let A, B, C, and D represent the vertices of the 8-in. square of paper. Folding on the dashed segments brings these vertices to positions directly "above" the midpoints of the creases, labeled A', B', C', and D'. We need to find the length of a side of $A'B'C'D'$. The process of folding has given us three

squares that form a geometric sequence in which the ratio of sides of consecutive squares is $\sqrt{2}:1$, or, equivalently, $2:\sqrt{2}$. If the original paper square is 8 in. on a side, the square formed by creases has side length $4\sqrt{2}$ .The distance between consecutive vertices of square $A'B'C'D'$ must be 4 in., or half the side length of the original square of paper.



Alternate solution: The sides of the squares are also the hypotenuses of isosceles right triangles. Use the Pythagorean theorem twice.

30. $16\sqrt{3}+80$ in.^{2}. One can begin with a square of any size. This solution uses an 8-in. square for the larger piece. When the 8-in. square is folded, four triangular "flaps" will be perpendicular to a square base. Each flap is an isosceles right triangle with legs 4 in.—half the length of the original square. The vertices of the right angles are 4 in. apart (see the March 29 solution), so the triangles that fit between the flaps are equilateral triangles with side length 4. The total area of four equilateral triangles plus the square attached to them is

$$A = 4\left(\frac{s^2\sqrt{3}}{4}\right)+s^2 = 16\sqrt{3}+16.$$

Adding the area of the 8-in. square gives a total surface area of $16\sqrt{3}+80$ in.^{2}.

31. $\frac{160\sqrt{2}}{3}$ in.^{3}. The prismatoid can be created from a rectangular prism by removing four triangular pyramids. The prism's height is the altitude to the hypotenuse of a right triangle with legs 4 in., so height $h = 4/\sqrt{2} = 2\sqrt{2}$ in. The prism's base is $(4\sqrt{2})^2 = 32$ in.^{2}, so the prism's volume is $64\sqrt{2}$ in.^{3}. The height of each pyramid is the height of the prism. The base of each pyramid is a right triangle with legs that are half the length of a base edge of the prism: $2\sqrt{2}$ in. Each pyramid's base area is

$$\frac{\left(2\sqrt{2}\right)^2}{2} = 4 \text{ in.}^2,$$

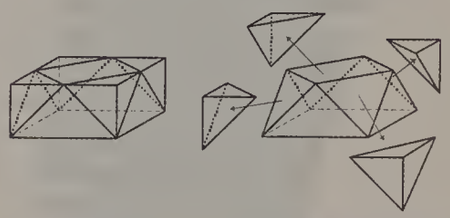
and its volume is

$$V = \frac{Bh}{3} = \frac{1}{3}\cdot 4\cdot 2\sqrt{2} = \frac{8\sqrt{2}}{3} \text{ in.}^3.$$

We have the following:

$$V_{prism} - 4V_{pyramid} = 64\sqrt{2} - 4\left(\frac{8\sqrt{2}}{3}\right) = \frac{160\sqrt{2}}{3} \text{ in.}^3$$

Remove four congruent triangular pyramids from this square prism.



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Call for Nominations
2013 Board of Directors Election

Each year, NCTM's Board of Directors makes important decisions that set the direction for the Council and mathematics education. The Board needs a broad representation of NCTM membership to benefit its discussions, inquiries, and decisions. In 2013, at least one high school teacher must be elected to ensure the balanced representation required by the bylaws.

NCTM has among its members many talented, energetic individuals who are qualified to assume leadership roles in the Council. The Nominations and Elections Committee needs your help in identifying these members by nominating them for Board Director positions.

Do you know someone who would bring valuable experience, perspective, and judgment to the Board? Nominate that person!

Would you be willing to serve on the NCTM Board of Directors? *You* can have a great impact on mathematics education at the national level.

For qualifications and responsibilities of NCTM directors and officers, school incentives for Board service, or to submit the names of nominees to the Nominations and Election Committee by completing the online nominations form go to: **www.nctm.org/nominations**. The chair of the committee will invite the nominee to complete an application after the first of the year.

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Intellectu and Other Mathem

What can American teachers learn about high-quality mathematics instruction from the Japanese teacher education process?

Blake E. Peterson, Douglas L. Corey, Benjamin M. Lewis, and Jared Bukarau

When mathematics teachers in the United States are asked what constitutes a high-quality mathematics lesson, the responses will likely vary greatly. As a mathematics teaching community, do we really have a clear idea of what high-quality instruction looks like? Jacobs and Morita (2002) attempted to answer this question when they asked American and Japanese teachers to comment on videotaped mathematics lessons taught in both countries. Their results suggest that American teachers view many strategies as acceptable; as a group, we do not have a clear picture of what best practice looks like. In contrast, Japanese teachers seem generally to agree on characteristics of high-quality instruction.

Japanese students' high scores on international comparisons and their teachers' highly rated lessons (Hiebert et al. 2003) led us to investigate how our Japanese colleagues perceive high-quality instruction. We did so by analyzing conversations between three cooperating teachers and three student teachers in a Japanese junior high school during a four-week student teaching experience.

The student teachers met with their cooperating teacher three or four times for a total of about ninety minutes to review each of the three lesson plans they wrote. Each lesson taught by a student teacher was observed by three to six other student teachers. At the end of the day, the student teacher who taught the lesson, the peers who observed the lesson, and the cooperating teacher participated in a meeting to reflect on the lesson. The main sources of data for our study were the recorded prelesson conversations between the student teacher and the cooperating teacher and the postlesson reflection meetings. We have organized and summarized the results of the analysis of our data as six principles of Japanese teachers' conception of high-quality mathematics instruction. (For a more detailed analysis, see Corey et al. [2010].)

1. THE INTELLECTUAL ENGAGEMENT PRINCIPLE

High-quality mathematics instruction intellectually engages students with important mathematics.

This principle appears to be the most central feature of a high-quality mathematics lesson. The five other principles are all closely tied to this central principle of intellectual engagement.

One example that illustrates this principle comes

al Engagement
Principles of
atics Instruction

from the first conversation between one of the cooperating teachers, Ueno, and Motori, a student teacher. The goal of this lesson was to help students understand the meaning of variables and notice the usefulness of variables in writing and solving general expressions. Motori was planning to use several questions relating to prices of items at a Japanese noodle restaurant. The first problem was, "If each egg costs 30 yen, then how much will one egg cost? How much will two eggs cost? How much will three eggs cost?" Eventually, Motori will ask the question, "How much will x number of eggs cost?"

In discussing this lesson, Ueno, the cooperating teacher, asks Motori to reflect on the nature of the questions that he is posing.

Ueno: I want you to think about this. Are there any places that students use their heads?
Motori: [*after a short pause*] There is no such place. Nothing at all.
Ueno: You are going to get to this point in the lesson, right? Students won't use their heads at all. I don't know what your plan in this part of the lesson is, but they won't use their heads here, either. This part only requires them to fill in a blank. I don't know if you really want to do that yet. You didn't plan to stimulate the students' "thinking processes." So it will become a shallow lesson. Do you think this [problem] will make students think?
Motori: No, I don't think this will make them think.
Ueno: What you are planning is just to kill the time … I think [what you are planning] is meaningless.

The cooperating teacher asks a simple question "Are there any places where students use their heads?" This one question summarizes this first principle well because if the answer is no, then there is no hope that the lesson will be a good one.

Along with emphasizing intellectual engagement of students, the conversations among the Japanese teachers focused on engaging students with important mathematics. In other instances, the cooperating teachers emphasized that the mathematics must never be sacrificed for a fun or interesting activity. This is not to say that these teachers do not try to make their lessons fun and interesting. We have evidence that they try hard to do just that. However, high-quality mathematics clearly requires mathematical work of a certain standard, even if it is not as much fun or as interesting as other possible activities.

Our data provide clear evidence that these cooperating teachers want students to be engaged in the lesson. Research on American teachers' conceptions of good instruction shows the same thing: Good teaching engages students (Wilson, Cooney, and Stinson 2005). However, when American teachers talk about engaging students, they usually refer to students engaging in active behavior—that is, moving around or working with their hands—or exhibiting other "on task" behavior (Wilson, Cooney, and Stinson 2005). Japanese teachers seem to emphasize going a step further to get students to do intellectual work, not just demonstrate engaged behavior.

2. THE GOAL PRINCIPLE

An ideal lesson is guided by an explicit and specific set of goals that consist of student motivation, student performance, and student understanding.

Each Japanese lesson plan contains goals for both the unit and the lesson. One interesting aspect of the lesson goals in these specific lesson plans is that almost every lesson includes at least one goal

concerning what students should understand and another concerning what students should be able to do, blending together two ideas that have often been set at odds in discussion in the American mathematics community. In addition, the lesson plans usually include a goal that addresses student motivation, attitude, or beliefs about mathematics.

One student teacher's lesson plan included the following goals for a lesson introducing the idea of variable:

- Students will understand the meaning of *variable*.
- Students will write a mathematical expression that pertains to the problem and that contains a variable based on the context of the problem.
- Students will develop an interest in using variables and will notice the usefulness of variables.

We were impressed by the student teachers' seriousness about these goals when preparing the rest of the lesson. Goal statements functioned as guides for their decisions about the lesson. At one point, cooperating teacher Ueno helped student teacher Motori realize that the task he had designed did not align well with the lesson's goal and offered some alternatives that better met the goal. The goals helped teachers craft the activities of a lesson that were focused and coherent but also helped them during the lesson.

3. THE FLOW PRINCIPLE

The flow of an ideal lesson stems from a question or a problem that students see as problematic. As students engage intellectually in the problem, building on their previous knowledge, they are supported in learning the lesson's big mathematical idea.

Of the six principles, flow is probably the most difficult to understand, primarily because it involves so many aspects of the lesson. At one level, flow deals with time allotment to different segments of the lesson and transitions between these sections. But flow is much more than keeping a lesson on pace and transitioning smoothly. A deeper notion of flow relates to the growth or development of the mathematical idea or ideas from closely related concepts that students already know and understand.

The selection of an appropriate problem, task, or activity is imperative for good pedagogical and mathematical flow. To emphasize this to Motori, Ueno said, "One thing you should do is to think about the flow of the human thinking process." The Japanese cooperating teachers use a phrase that is translated as "natural path" or "natural way" in reference to the way in which the lesson would help students think about the mathematics. The naturalness has to do with what seems natural to the student, not necessarily the logical way that mathematics is structured in textbooks or organized by the discipline of mathematics.

Another characteristic of the flow of a lesson is how student thinking is used. Not only should the problem be accessible to the student at his or her current level of mathematical understanding; it should also build toward subsequent mathematical ideas in the unit. Thus, the teacher must be prepared and willing to use emerging student ideas to move the mathematical flow forward. The concept of pedagogical flow is what these cooperating teachers use to craft their mathematics lessons. It builds on students' current knowledge and ideas and leads them in a natural way to the learning objectives.

4. THE UNIT PRINCIPLE

A lesson is created in the framework of past and future lessons, particularly between lessons in a unit but also between units and grades. The lessons in a unit help students progress to ways of thinking, writing, and representing mathematics evident in the discipline of mathematics.

From the conversations that we analyzed, it is clear that a high-quality lesson is best understood in the context of past and future lessons. A common Japanese lesson plan template includes information about the teacher's class, a list of goal statements for the unit, and other information about the unit—for example, a list of the lessons that will be taught. The cover page of each lesson plan describes the context of the preceding and following lessons; thus, each lesson is situated in the lessons that surround it.

The importance of seeing each lesson in the context of the lessons that surround it is evident in the following interchange between Ueno, the cooperating teacher, and Motori, the student teacher.

Ueno: When you become a real teacher, what would you teach next? Which path would you follow in your lesson?

Motori: In the next lesson?

Ueno: When you become a real teacher, you won't teach only this lesson, right? You have 365 days in a year, from seventh grade to ninth grade. If you are going to teach math classes well, you have to have an image of what you would do next in your mind. What would you do?

Having a clear idea of "what you would do next" can greatly influence how teachers approach and facilitate a lesson. For teachers, having a clear direction in mind for how to help students explore the mathematical terrain seems to be as important as knowing where students have been.

5. THE ADAPTIVE INSTRUCTION PRINCIPLE

High-quality instruction adapts so that all students are engaged in mathematical work that appropriately challenges their current understanding.

Japanese cooperating teachers adapt lessons by considering two categories of students—those who understand and those who do not—and by ensuring that all students are intellectually challenged during the lesson, even those who appear to understand the material easily. In considering both types of students, teachers are expected to adapt their instruction so that all students are challenged at their level. The teachers prepare beforehand how to adapt the task or extend the problem to keep all students challenged.

Japanese teachers rarely organize a class in groups according to ability. Instead, students are often divided into groups of four, not to homogenize abilities but to ensure that each group consists of students with different strengths, talents, and mathematical ability. Thus, the two categories of students are not physically separated. All students are given the same tasks and are expected to complete the same assignments.

The students in the two categories vary from day to day. The students who need help today may need an extra challenge tomorrow. To know which students need more of a challenge and which are struggling, teachers check on students and evaluate their work by walking around the class and checking on groups (if the students are organized in groups) or individual students. Sometimes, to keep all students challenged, the teacher will assign a few problems, at least one of which presents a challenge. Thus, the teacher can spend more time working with those who are struggling.

A second student teacher, Tomoko, prepared a lesson on solving systems of linear equations with two unknowns. Before the lesson, Tomoko had prepared some questions for students who might struggle (and thus need a hint or help) and other questions for students who might easily solve the problem and could benefit from being further challenged. Tomoko did not know beforehand which students might need help or which students would need to be pushed with a more challenging task. As Tomoko taught, she realized that the questions she had prepared did not help move students toward solutions.

Tomoko shared with her cooperating teacher her difficulty in challenging both those students who could easily solve the problem and those who had struggled:

> ***Tomoko:*** There were some students [who] solved the problem very quickly, and on the other hand there were some students who couldn't solve it at all. I wasn't able to follow up on those two types of students. Now I can, but at that time I wasn't sure what should have been said. Nobody was able to come up with all four methods, but to those students or groups [who] used two or three methods, I said, "Are there any other ways?" or "How would elementary school students solve this problem?" But there was little reaction to those questions, and I wonder if my questions weren't appropriate.

Tomoko saw this as a critical shortcoming of her lesson. This was the first specific point that she mentioned in discussing the lesson with her cooperating teacher. She went on to emphasize that she could have overcome this shortcoming with better preparation. Preparing questions beforehand for students who solve the task quickly or students who struggle is clearly an important principle that Tomoko had been taught and a principle that could benefit all teachers.

6. THE PREPARATION PRINCIPLE

High-quality instruction requires a well-thought-out, detailed plan that addresses the previous five principles and ties them together in a coherent lesson.

The desire to create a lesson that satisfies all six principles is what drives Japanese teachers' preparation. Altering the wording of a question or considering how to handle possible student questions may not make a distinct difference, but when a teacher prepares for all contingencies, the lesson as a whole emerges as more coherent, engaging, and meaningful for students. This preparation not only helps teachers know what to do in most situations but also helps maintain good lesson flow and enables students to reach the learning goal for that lesson.

The importance and role of preparation for these Japanese teachers were illustrated during the discussions that occurred after each lesson. The topics discussed in the postlesson conversations focused mainly on implementation and adequate preparation. Many of the problems in implementing the lesson plan effectively were attributed to lack of proper preparation. The attitude that nearly all problems during the lesson can be taken care of with proper planning dominated postlesson discussions. One cooperating teacher emphasized to a student teacher that without preparation, a teacher will not be able to handle every situation that arises during the lesson.

GUIDING QUESTIONS

These six principles can be implemented in a variety of effective instructional strategies and methods. These principles seem to provide a solid basis for high-quality instruction, although clearly they do not capture everything that is important in a lesson. We suggest some ways in which knowledge of these principles could assist teachers in improving

the quality of their lessons. Several questions that teachers can ask themselves as they prepare lessons follow. The questions are organized by principle.

1. The Intellectual Engagement Principle

- Where in the lesson will students think? That is, are students asked to try to solve a problem in which the solution process is not immediately apparent?
- What important mathematical work will students be doing during this lesson?
- Is the lesson as lively and interesting as possible without sacrificing important mathematics?

2. The Goal Principle

- What are the specific mathematical goals that I want students to learn?
- How will my lesson develop both an understanding of mathematical ideas and an understanding of mathematical skills?

3. The Flow Principle

- What problem, question, or dilemma do the mathematical ideas naturally develop from?
- How does the lesson help students see a need to resolve the problem, question, or dilemma?
- What are possible student responses to the major tasks or questions? How will they be used or addressed in the lesson?

4. The Unit Principle

- How will this lesson naturally connect to ideas in previous lessons?
- How will this lesson build a foundation for future lessons?

5. The Adaptive Instruction Principle

- What scaffolding have I prepared for the students who will need help today?
- How will I challenge the students who finish quickly?

6. The Preparation Principle

- How well would my colleagues evaluate my lesson from the prepared lesson plan?
- Have I adequately planned the specific wording of the questions I intend to ask?

CONCLUSION

We have characterized six common aspects of Japanese mathematics teaching as principles of high-quality instruction. The flow and unit principles relate to the coherence and connections of ideas within and between lessons. The goal principle calls for clear outcomes for instruction that help guide teachers' decisions. The intellectual engagement principle and the adaptive instruction principle seek to ensure that all students think about, use, and develop important mathematical ideas. To implement these principles with a high degree of fidelity takes a well-thought-out lesson plan, stipulated in the preparation principle.

These six principles do not answer all questions about effective instruction. However, they do suggest what we can focus on while preparing or evaluating lessons to improve mathematics instruction.

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CALL FOR MANUSCRIPTS

2014 Focus Issue

What Makes Ideas Stick?

Teachers design lessons to engage students with big ideas of mathematics. Does engagement necessarily imply that students will retain what we hope they will retain? How do students in your classes develop an appreciation for mathematical big ideas and the relationships among them? What lesson characteristics seem to promote the permanence of concepts in the minds of your students?

What works? Write about your big ideas—*big* big ideas or *small* big ideas—that help make mathematics stick. Following are a few suggestive, but not prescriptive, prompts:

- Great lessons stay in the minds of students. What lessons do you have that engage your students in exciting and meaningful mathematical experiences?
- Some ideas reside with students because of the instruction that introduces them more than the inherent mathematics. Do you start with an intriguing idea and then develop it throughout class, or do the big ideas build in the background?
- Students often surprise us with their own "aha!" moments, usually as they experience the epiphany of an unexpected connection. What are these moments, and how do we foster these connections and epiphanies?
- How do reasoning and sense making evolve to foster mathematical memory? What are high-leverage practices that help ideas stick?
- What kinds of challenges do big mathematical ideas face in the current political climate?
- How can technology impact the "stickiness" of ideas?
- How do we ensure that students internalize the mathematics that they need to be successful in subsequent courses? How do they develop the mathematical sophistication to adapt to an ever-changing world?

Contribute to this special *MT* issue by highlighting mathematical and pedagogical big ideas from your own teaching. Show us what helps students retain ideas and what leads them to an appreciation of broad concepts.

The *MT* Editorial Panel invites submissions for the 2014 Focus Issue. Please submit manuscripts at **mt.msubmit.net** by **May 1, 2013.** Be sure to check the box indicating that this manuscript is for the 2014 Focus Issue entitled What Makes Ideas Stick? Guidelines for the preparation of manuscripts can be found at www.nctm.org/publications/content.aspx?id=22602. No author identification should appear in the manuscript's article file. If you have ideas related to this topic and would like to discuss them before sending a manuscript, please contact Albert Goetz, agoetz@nctm.org.

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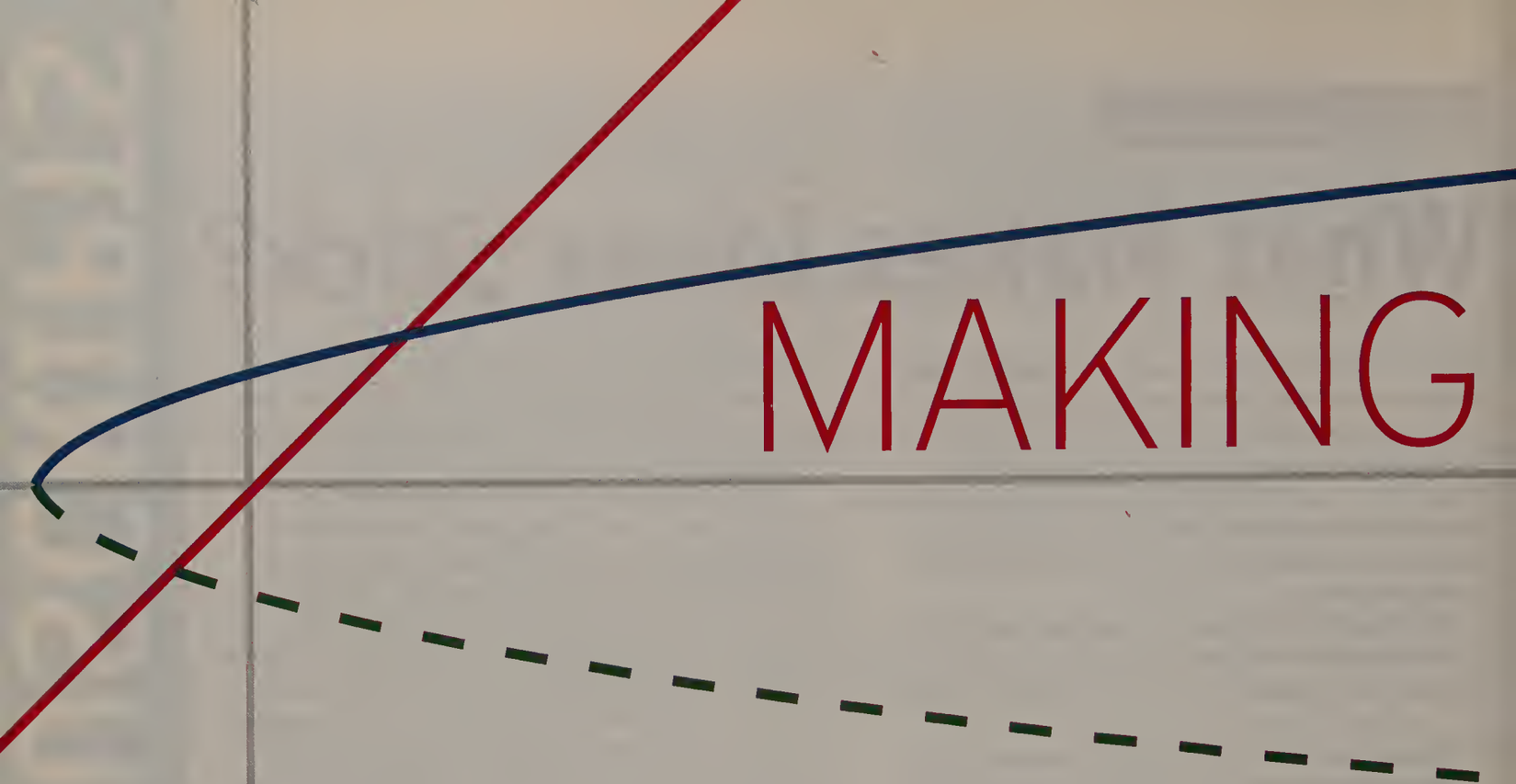

MAKING

EXTRANEOUS

Do you always have to check your answe

"Always check your answers when solving any radical equation." While observing a class as a mentor, I overheard this response from a student teacher to a student's question, "Do we need to check our answers to this [cube root] equation?" Good teachers will always emphasize checking work after solving a problem, but this particular question referred to the possible introduction of an extraneous root. The solution of a radical equation involving only cube roots, where checking for extraneous algebraic solutions is not required, was not discussed by the student teacher; there was no discussion or examination of function domain. The underlying reasoning and sense making of extraneous solutions did not occur at any point in the lesson. As a result, I began to develop a technology-based lesson using the TI-Nspire™ CAS.

Principles and Standards for School Mathematics (NCTM 2000) states, "Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning" (p. 11). The lesson in which I observed the teacher-student exchange quoted above took place in a second-year algebra classroom. It focused solely on procedures and used technology simply as a computational tool for checking answers to equations containing square and cube roots. The focus on reasoning and sense making with technology in the lesson presented here will enable students to do more than just carry out procedures; they will be able to understand the procedures and know how

SENSE OF SOLUTIONS

hen solving a radical equation?

Jeremy S. Zelkowski

the procedures might be used in different situations while interpreting the results (NCTM 2009).

The lesson includes three key elements of reasoning and sense making with functions: (1) using multiple representations of functions as a means of demonstrating mathematical flexibility in problem solving; (2) modeling by using families of functions; and (3) analyzing the effects of parameters (NCTM 2010). According to the Programme for International Student Assessment (PISA 2007), American students lag behind international students in their ability to analyze, reason, communicate, solve, and interpret a variety of mathematical problems. The following lesson allows teachers to use a pedagogical approach to teaching mathematics through problem solving (see Schoen 2003) and also address these PISA areas of concern.

By exploring the following two problems, students will improve their ability to understand and solve problems involving radicals:

Problem 1: Determine an equation with a single radical that, when solved algebraically, will yield one unique real solution and two extraneous solutions.

Problem 2: Without working out this problem with paper and pencil, predict how many real and extraneous solutions would be obtained when solving $\sqrt{x+3} - 2 = 2x + 3$ algebraically.

For each problem, explain your reasoning in support of your responses.

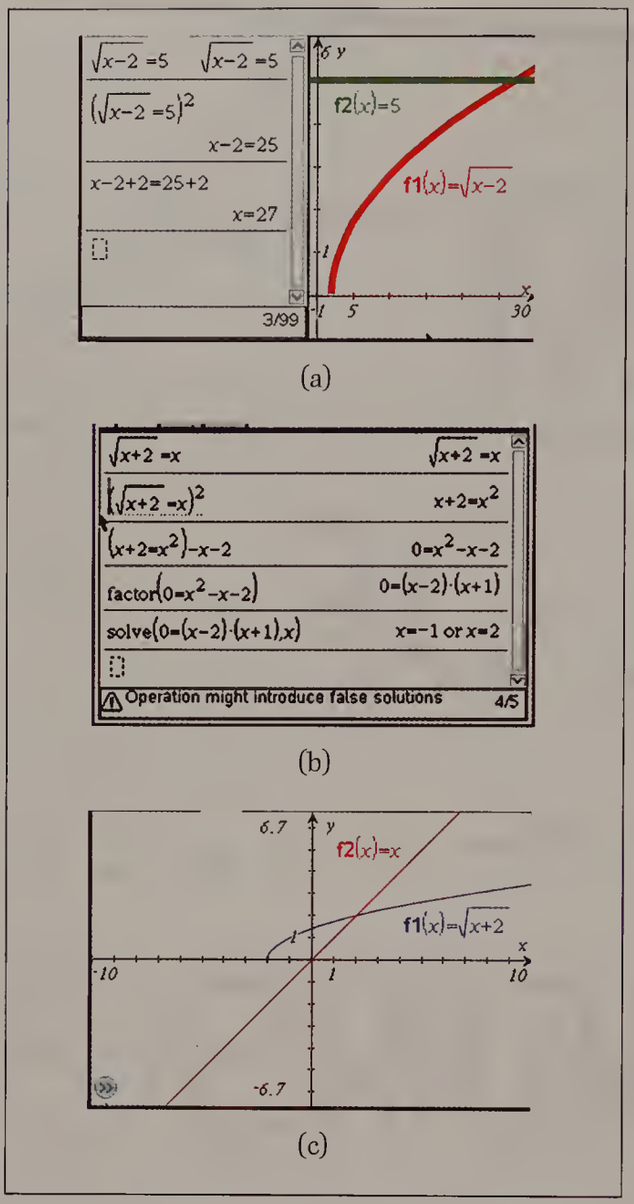

Fig. 1 The power of the graphical solutions (**a** and **c**) and the CAS solution (**b**) to equations 1 and 2 adds to student understanding and establishes new connections.

SOLVING PROBLEM 1

At the start of the lesson, the teacher and students can work through skill-focused problems similar to equations 1 and 2, as needed.

Equation 1: $\sqrt{x-2} = 5$
Equation 2: $\sqrt{x+2} = x$

Then students can begin to make connections between solution methods by using paper and pencil or a computer algebra system or by creating a graph. Alternatively, the teacher may elect to begin the TI-Nspire lesson through discovery. The CAS eliminates computational errors and allows students to focus on content and connections between multiple representations without losing the opportunity to demonstrate problem-solving ability and algebra skills (see **fig. 1** for graphical and CAS solutions).

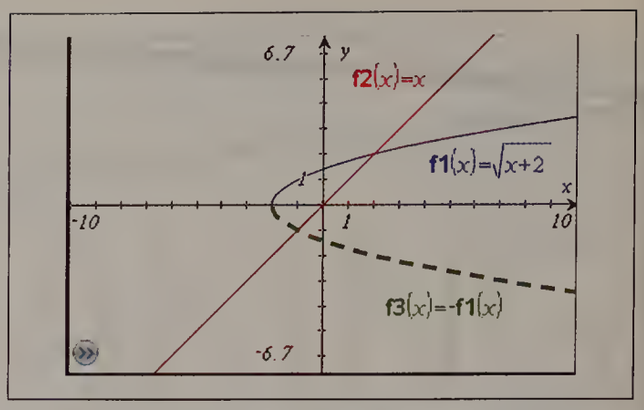

Fig. 2 Graphing the negation of the radical function reveals the extraneous algebraic solution.

When students demonstrate the algebraic solutions on the TI-Nspire CAS, the handheld tool gives this warning: **Operation might introduce false solutions**, a signal for students to explore further.

In **figure 1c**, the graph indicates that equation 2 has only one real solution, $x = 2$; thus, the other algebraic solution ($x = -1$) must be extraneous. Students should explore how the graph can be used to explain the extraneous solution, which is visible when $-f1(x)$ is graphed and an intersection point occurs at $(-1, -1)$ (see **fig. 2**). Hence, the $x = -1$ extraneous solution now appears graphically.

Students must learn that squaring both sides of the equation, even one as simple as $x = 2$, will introduce extraneous roots. They need to realize that squaring to undo the square-root function means that both cases of the radical function need to be considered. For example, in the equation $\pm\sqrt{x+2} = x$, both

$$\left(\sqrt{(x+2)}\right)^2 = x^2 \text{ and } \left(-\sqrt{(x+2)}\right)^2 = x^2$$

result in $x + 2 = x^2 \rightarrow x^2 - x - 2 = 0$.

EXAMINING FAMILIES OF EQUATIONS DYNAMICALLY

The interactive geometry capability of the TI-Nspire (with or without CAS) can take the lesson to a higher cognitive level. As **figure 3** shows, students are able to manipulate the linear function $f1(x)$ dynamically. They can manipulate the y-intercept and slope independently. To manipulate the y-intercept, students move the cursor arrow near the y-intercept (see the crosshairs near the origin in **fig. 3a**) and grab the line (center of touchpad). To change the slope, students move the cursor arrow near the ends of the line (see **fig. 3b**, lower right) and grab.

The parameters a, b, and c of the radical function can be changed incrementally. **Figure 3** shows a student's work in shifting the radical function, $f2(x)$, to the right three units, thus eliminating the

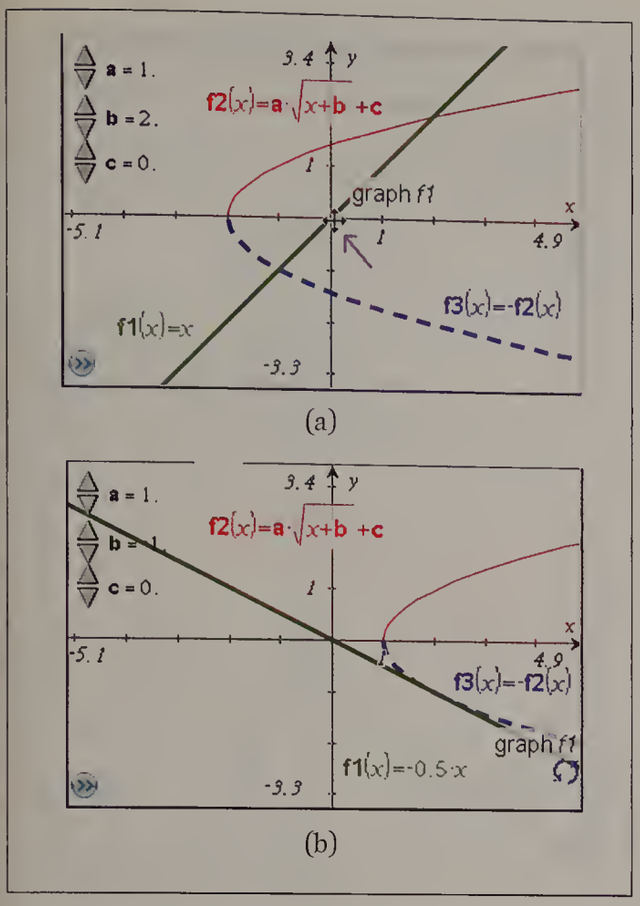

Fig. 3 A student changes parameters to transform both the radical and the linear function. Going from (**a**) to (**b**), the slope of the line and the x-intercept of the radical function have been adjusted.

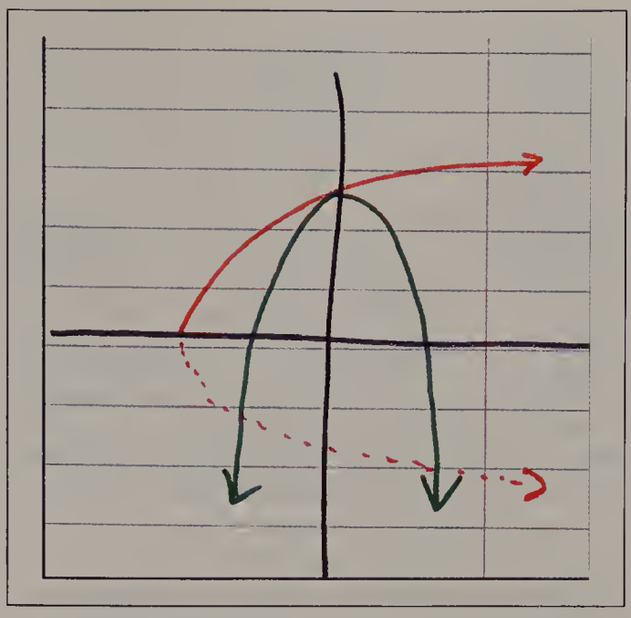

Fig. 4 This student initially attempted to solve problem 1 by graphing a radical and a quadratic function. The parabola is tangent to the radical function and intersects the negation of the radical function twice.

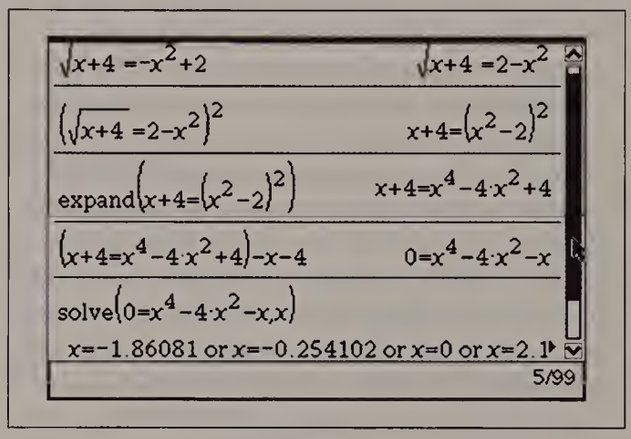

Fig. 5 A student's algebraic solution to the proposed graphical solution seen in **figure 4** uses CAS.

intersection with the linear function. The student also changed the slope of $f1(x)$ from 1 to –0.5. These transformations produce two functions that, when set equal to each other, will produce an equation with two extraneous algebraic solutions. The dynamic capability empowers students to build their own mathematical understanding as well as to exercise their reasoning and sense making.

However, the possibility of complex solutions of equations from the family of functions generated by $f1(x)$ and $f2(x)$ in **figure 3** can easily be overlooked by students. Teachers need to be ready to incorporate guiding questions such as these: What additional possibilities exist for solution sets that we have not yet examined graphically? What are the graphical representations for each combination of solutions? Are there possibilities that we may be overlooking? What would happen if $f1(x)$, $f2(x)$, and $-f2(x)$ do not intersect at all?

After discussing the complex solution possibilities, students should begin to think about the necessary conditions that will add a third possible solution, as required by problem 1 of the lesson. The solution sets that can be generated by the family of functions explored in **figure 3** have only two algebraic solutions (real, extraneous, or complex). This will help differentiate the differences between extraneous and complex solutions—which students sometimes think are the same thing.

When I have used this lesson, most students gravitate toward sketching a graphical solution for problem 1. At times, students use only polynomial functions and try a linear function composed with the basic radical function. This approach limits the initial examination.

Without an equation, students have trouble making a start on this problem. Trial and error or guess and check usually do not yield a solution. One student proposed a possible graphical solution (see **fig. 4**), an equation that would involve the radical function previously explored and a quadratic function. This choice shows great graphical intuition, but this solution needs further examination. The intersection near or on the y-axis must be at a point of tangency. This proposed solution leads to one piece of mathematical understanding needed for a solution.

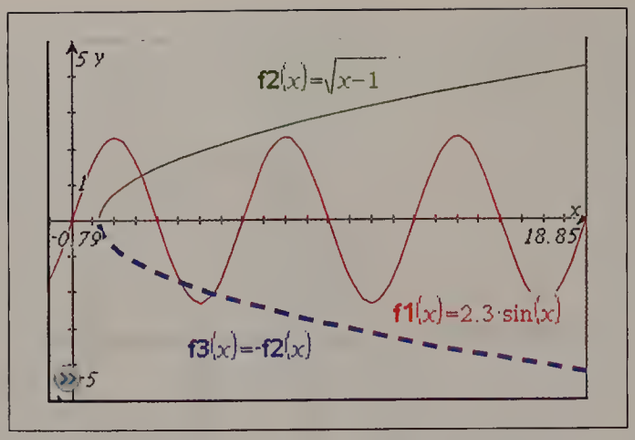

Fig. 6 Students' only recourse to solving an equation such as $2.3 \sin x = \sqrt{x - 1}$ is to use a graph or a CAS.

Students can then explore this proposed solution by solving an equation (see the CAS solution shown in **fig. 5**). By using the CAS to solve this proposed equation algebraically, students realize that four answers result: $x \approx -1.86$, -0.25, 0, and 2.11. The point (in **fig. 4**) near (0, 2) is worthy of discussion because it ended up not being a point of tangency. The two functions intersect twice (at $x = 0$ and at $x \approx -0.25$) in the region near the positive y-axis.

Students need to make two important discoveries in the lesson to be able to solve problem 1. First, they must realize that an equation with a single cube-root radical will not generate any extraneous solutions; only an even-root radical will. (A fuller discussion of this concept is available at the website provided at the end of the article.) Second, students must discover that after they square both sides of the equation, the result must generate three solutions. In other words, squaring the desired equation must produce a cubic polynomial.

Once students make the mathematical connections needed, they can use the TI-Nspire capabilities (CAS, parameterization, transformation of linear functions) to generate a correct solution to problem 1. For example, $\sqrt{x^3 - 4x} = x - 0.5$ yields one solution, but there are an infinite number of answers involving any radical with an even index. At this point in the lesson, teachers could ask questions such as these:

- What would happen if we try to generate a solution that includes a trigonometric or a transcendental function?
- What would happen if we try to generate an equation with only integer or rational solutions?

One student used the TI-Nspire to generate a fairly convincing graphical argument to a trigonometric-transcendental question (see **fig. 6**).

Some students have tried to generate the point of tangency seen in **figure 4** or integer-only solutions by reversing the problem—that is, by starting with

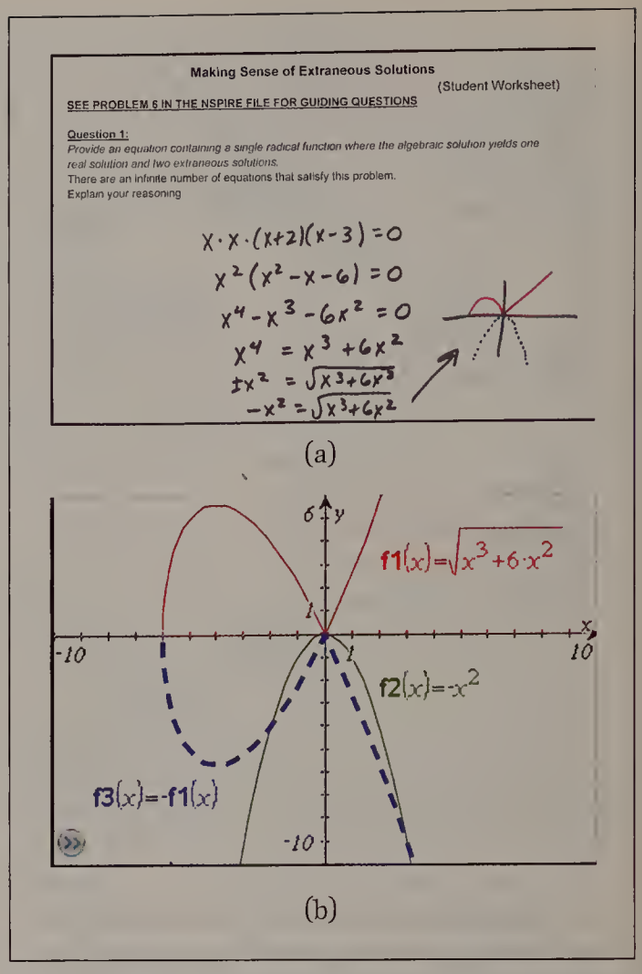

Fig. 7 A student uses reversibility to generate a solution to problem 1 (**a**) and then verifies the solution graphically (**b**).

a factored form of the equation and working backward algebraically. For example, one student started with $x \cdot x \cdot (x + 2) \cdot (x - 3) = 0$, which has a double root at $x = 0$ and could be a solution to the graphical representation seen in **figure 4**. The student first removed parentheses and then ended up with the equation $\sqrt{x^3 + 6x^2} = -x^2$ (see **fig. 7**). An examination of the graphical representation made clear that the x^2 should be on the right side of the equation and should be negative. Although this double root is not a point of tangency as defined in geometry or calculus, the student found a correct solution to problem 1 with integer-only algebraic solutions.

This lesson thus helps develop students' ability to use mathematical flexibility and reversibility in problem solving (Krutetskii 1976; see also Rachlin 1985). Making connections between algebraic and graphical solutions also demonstrates the use of mathematical flexibility in problem solving. The family of equations of the form $a\sqrt{x + b} + c = mx + B$ cannot be used to solve problem 1. Deep understanding of the mathematical connections between algebraic and graphical solutions from this family of functions presents an environment conducive to reasoning and sense making. Both are needed to move to mathematical problems requiring higher levels of cognitive effort.

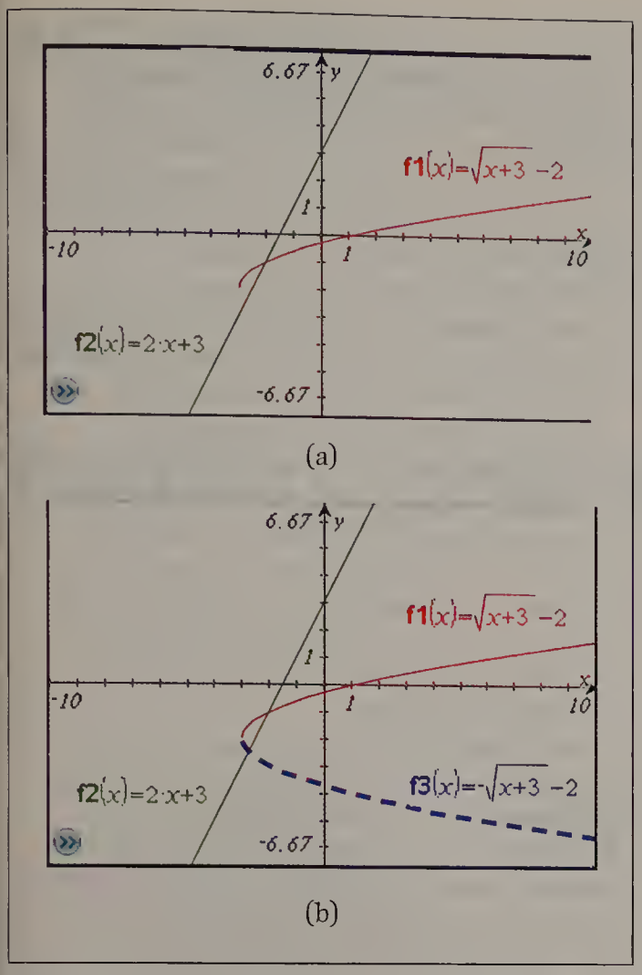

Fig. 8 Students can solve problem 2 using a graph (**a**); the other branch of the sideways parabola shows an extraneous root (**b**).

SOLVING PROBLEM 2

The beauty of this problem is that it allows for different approaches and different solutions. The previous class discussions, leading questions, and equations explore reflecting the simple square-root function over the x-axis, thus creating a parabola opening to the right or left with its vertex on the x-axis. The lesson does not guide students through the examination of a square-root function that crosses the x-axis, as is the case with problem 2, or one that lies completely above the x-axis. If we plot the left side of the equation in problem 2 or visualize $y = \sqrt{x}$ shifted to the left three units and down two units, we see that this radical function crosses the x-axis (see **fig. 8**). Student responses to this problem have generally fit into one of two approaches.

Most students' first response includes shifting the radical and the linear functions up two units. Essentially, this is what we would expect or want students to do first algebraically—add 2 to both sides of the equation first, then square, and so on. Thus, students accurately predict one real solution and one extraneous solution.

The second type of response generally comes from creating the parabola seen in **figure 8b**. I want students to examine and understand—algebraically and graphically—what will happen if we square both sides of the equation first. Doing so is a common first step by some students when they see a square root sign. Mathematically, squaring first is not as efficient. Only a few students have ever submitted an answer such as that shown in **figure 9**. This solution accurately depicts squaring both sides first and eventually solving a quartic polynomial equation that will generate a real solution and three extraneous solutions.

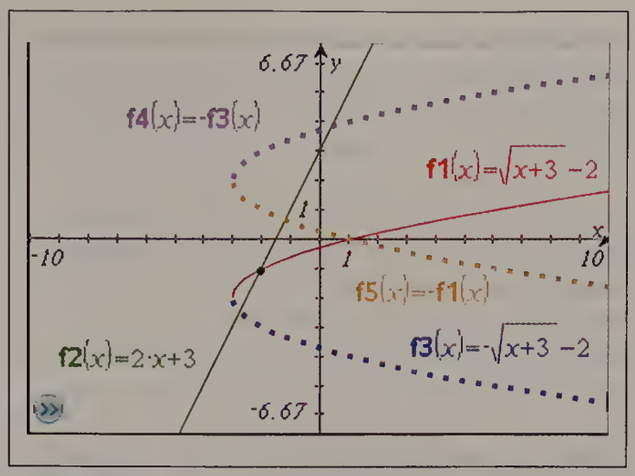

Fig. 9 The graphical representation presents four solutions to problem 2, three of which are extraneous.

Students must consider the four cases of the radical equation $\pm[\pm\sqrt{x+3} - 2] = 2x + 3$: (1) $\sqrt{x+3} - 2 = 2x + 3$; (2) $-\sqrt{x+3} - 2 = 2x + 3$; (3) $-[\sqrt{x+3}] - 2] = 2x + 3$; and (4) $-[-\sqrt{x+3} - 2] = 2x + 3$. Cases 1 and 3 yield the equation $-4\sqrt{x+3} = 4x^2 + 11x + 2$, and cases 2 and 4 yield the equation $4\sqrt{x+3} = 4x^2 + 11x + 2$. We now solve $\pm 4\sqrt{x+3} = 4x^2 + 11x + 2$, which both produce the quartic polynomial equation $16(x + 3) = 16x^4 + 88x^3 + 137x^2 + 44x + 4$, with solutions at $x = -2.75, -2, (-\sqrt{41} - 3)/8 \approx -1.1754$, and $(\sqrt{41} - 3)/8 \approx 0.4254$.

In **figure 9**, the highlighted point depicts the only real solution at $x = -2$ and represents case 1, $f1(x) = f2(x)$. The three extraneous solutions are depicted with the dotted functions intersecting the linear function. Case 2, $f3(x) = f2(x)$, yields $x = -2.75$. Case 3, $f5(x) = f2(x)$, yields $x \approx -1.1754$. Case 4, $f4(x) = f2(x)$, yields $x \approx 0.4254$.

By not providing a direct path or direction to solve problem 2, I am able to assess students on a higher cognitive level (Stein et al. 2000).Thus, I do not provide an in-depth exploration of this problem type during the actual lesson. This approach also serves as a way to create discussion about the mathematics of this lesson after the assignment has been collected.

DEVELOPING TPACK

This lesson was originally planned for secondary school mathematics preservice teachers and their

development of Technological Pedagogical and Content Knowledge (TPACK; see Mishra and Koehler 2006), mathematical reversibility, flexibility, and reasoning and sense making.

It has generally been acknowledged that it takes three to five years of training in technology for teachers to be comfortable and knowledgeable enough to use it regularly in teaching (Dwyer, Ringstaff, and Sandholtz 1991; Means and Olson 1994). That was true for a different generation of teachers. In today's technologically driven world, three to five years is too long. In theory, because today's generation of secondary school mathematics teachers grew up with technology in their hands often and early, this time should be cut in half (see Leatham 2007; Norton, McRobbie, and Cooper 2000; Zelkowski 2011).

This lesson is just one that can help in the development of TPACK as well as content knowledge. Moreover, this approach to a topic traditionally taught through rote procedure can be extremely engaging in the high school classroom.

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Editor's note: The student activity sheet, teacher notes, and TI-Nspire lesson file can be downloaded from the author's website: http://sites.google.com/site/jszelkowski/Home/mtextraneous.

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Michael Dempsey

Dealing Cards with Confidence

Activities for Students appears six times each year in *Mathematics Teacher*, often providing in reproducible formats activity sheets that teachers can adapt for use in their own classroom. Manuscripts for the department should be submitted via http://mt.msubmit.net. For more information on the department and guidelines for submitting a manuscript, please visit http://www.nctm.org/publications/content.aspx?id=10440#activities.

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When understood and applied appropriately, mathematics is both beautiful and powerful. As a result, students are sometimes tempted to extend that power beyond appropriate limits. In teaching statistics at both the high school and college level, I have found that one of students' biggest struggles is applying their understanding of probability to make appropriate inferences.

Whenever I articulate my conclusion to a test of significance, I always emphasize the phrase "We have evidence to support...." Yet many students will use language such as "These results prove ...," even though I never make any such statements. I underscore that unlike geometry, in which we can prove that two triangles are congruent, statistical inference is about making informed decisions on the basis of what is likely or unlikely. We make the best possible decision according to what the data suggest, but we are not *proving* anything.

This hands-on task meets students at their understanding of sampling distributions and the central limit theorem. Then we use the concrete and familiar example of the card game Hearts to investigate whether the teacher is cheating. The activity leads students to a thorough and complete understanding of the appropriate use and interpretation of a confidence interval as an estimate of a parameter.

Students' prior experience with card games will help them focus on the key concepts of probability and inference without becoming mired in scientific details. The activity also uses the graphing calculator to perform a simulation that would otherwise be too tedious to be practical in a classroom setting. The College Board's description of the AP Statistics course content states: "The AP Statistics course depends heavily on the availability of technology suitable for the interactive, investigative aspects of data analysis" (p. 6).

The activity begins with students passing around an unfair deck of cards that the teacher has loaded before class. The deck can be loaded in the following way so that only two decks are needed: remove three cards of each suit (diamonds, spades, or clubs) from the deck, replacing them with their equivalent value of a heart.

The teacher tells the class that he has been accused of cheating and needs their help to prove his innocence. As the deck circulates around the room, each student shuffles it and draws a card, notes its suit, and replaces the card in the deck. A student or the teacher will keep a tally on the board of how many cards of each suit were drawn until 40 cards are drawn; this will constitute a single $\hat{p}$ estimate of the proportion of hearts in the deck.

Next, each student in the class will use the graphing calculator to simulate

drawing 40 cards from a fair deck. Each student should complete at least one simulation, but the more the better. Students use the probability simulation application that is found on any TI83/84 Plus calculator. They will record only the suit of the card drawn. After students have drawn 40 cards, they will record the sample proportion, $\hat{p}$, for the proportion of cards that were hearts.

Next, students review their previous knowledge of sampling distributions and the central limit theorem. These topics should be fresh in their minds, and they should quickly be able to describe the center, shape, and spread of the sampling distribution of $\hat{p}$. Because students are unfamiliar with the concept of inference about an unknown parameter at this point, they will assume that using the formula

$$\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$$

is appropriate. The teacher should emphasize that in this example, based on drawing from a fair deck, p is known to be .25. Therefore this formula cannot be used if we are using the confidence interval as an inference tool. Instead, we must use

$$\sigma_{\hat{p}} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$

Next, we review the previously studied topic of the normal probability distribution to explain why we add and subtract 1.96 standard deviations to produce the confidence interval.

Students are now ready to construct the interval and make an informed decision, based on collected data, as to whether or not the interval contains the parameter. For this reason, more simulations are preferred. The more samples that students produce in this activity, the more likely they are to encounter intervals that do not contain p.

Another way to accomplish this would be to use a 90% confidence interval. This activity uses 95% for the sake of student familiarity with the 68-95-99.7% rule of thumb. The great misconception about confidence intervals that this activity is designed to avoid or help clarify is that students believe that this 95% confidence interval has a .95 probability of containing p. The error, although subtle, can be damaging to a correct understanding of statistical inference.

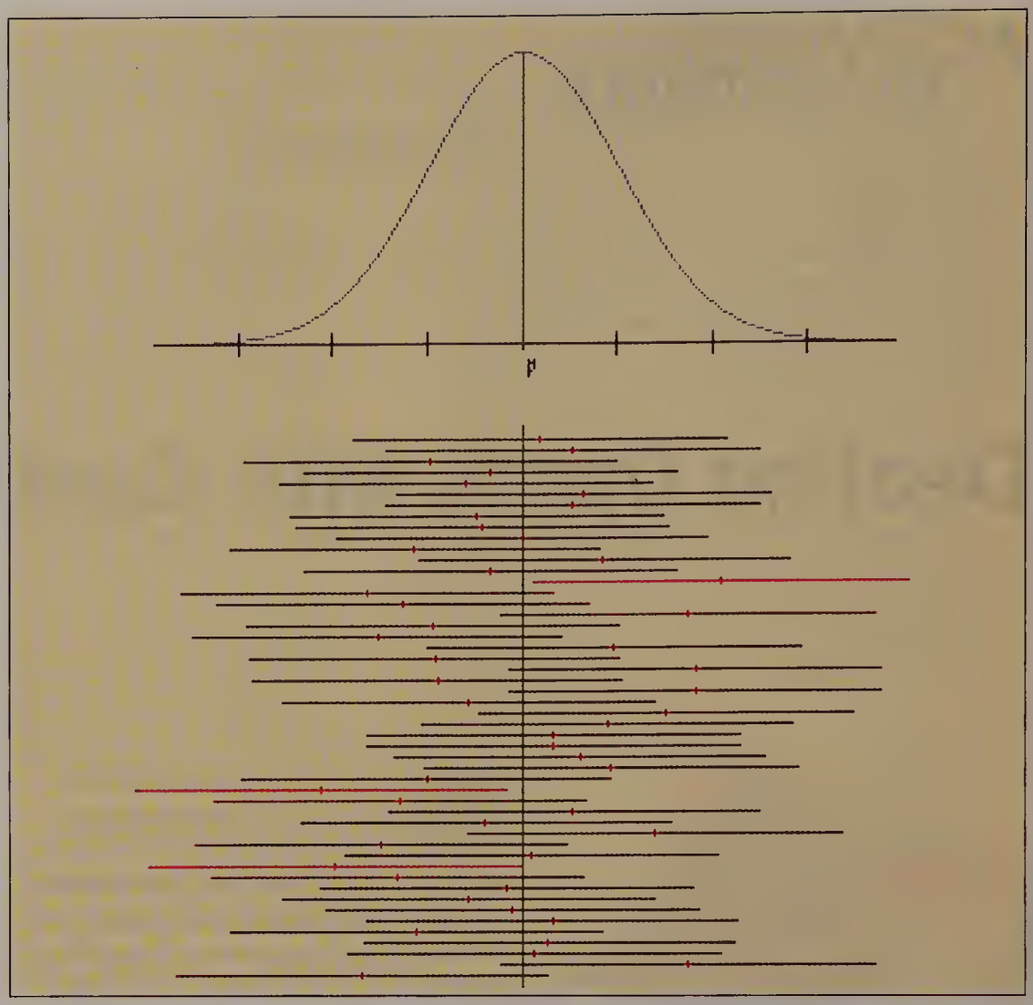
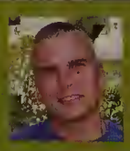


Fig. 1 In this example of 50 95% confidence intervals computed from 50 different samples, 3 of the 50 do not contain the true value of the parameter.

This interpretation places all the emphasis on this one interval, whereas the emphasis should remain on the parameter and what the data suggest about the parameter. Suggesting a 95% probability that the interval contains the parameter may make it seem as if p wanders about and lands in this one interval 95 out of 100 times; it is really the interval that is "wandering." This error is similar to jumping to the conclusion that a significance test proves that a hypothesis is correct, rather than providing evidence in support of the hypothesis.

A sketch of a sampling distribution for $\hat{p}$ on the board with a vertical line below the parameter p will help lock in this idea. Then the students draw their confidence intervals and examine how many contained p (see **fig. 1**).

The activity concludes with an explanation of how to use the graphing calculator to compute the confidence interval. This explanation gives students a more efficient tool for computing a confidence interval; allowing them to focus on the correct interpretation and application of a confidence interval. Students will have the chance to compare their computations with the results provided by the calculator.



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Activity: Exploring Confidence Intervals

Part 1: Investigation and Simulation

Mr. Dempsey has been accused of cheating in the annual faculty Hearts tournament. One teacher claims that 40% of the cards in Mr. Dempsey's deck are hearts. Mr. Dempsey counters that his deck is just "lucky" (he's always played with it) and that other faculty members just don't understand "random chance." He needs your help to prove his innocence and be allowed back into the tournament. We will compare Mr. Dempsey's deck to many fair decks to answer the question: Is Mr. Dempsey's lucky deck also a fair deck?

Record the results of 40 cards drawn with replacement from Mr. Dempsey's "lucky" deck below:

1. If a deck is fair, the proportion of hearts, p, is known to be .25.
2. Does this mean that every sample of 40 cards will produce 10 hearts?
3. How many hearts in a sample of 40 would you consider too few to imply a fair deck?
4. What proportion is this?
5. How many hearts would you estimate as too many for a sample of 40?
6. What proportion is this?

Part 2: Simulation

Next, we will use the calculator to simulate selecting cards from a standard deck of 52 cards with replacement.

1. Press the **APPS** key on your graphing calculator; then select **Prob Sim**.
2. Under **Probability Simulations**, select 5: **Draw Cards**.
3. Use the simulator to draw 40 cards. Use the table below to tally and total the number of cards from each suit. Complete a total of 5 simulations.

Part 3: Review of Sampling Distributions

(Note: $\hat{p}_D$ represents the sample proportion that you recorded from Mr. Dempsey's deck; $\hat{p}_S$ represents the sample proportion recorded from the calculator simulation.)

Mr. Dempsey's Lucky Deck			
Hearts	**Clubs**	**Diamonds**	**Spades**
Total =	Total =	Total =	Total =

1. What was the sample proportion of hearts in the sample taken from Mr. Dempsey's lucky deck?
   $\hat{p}_D =$_____

2. What were the sample proportions of hearts in your sample of 40 cards taken from each calculator simulation?
   $\hat{p}_{S_1} =$_____ $p_{S_2} =$_____ $\hat{p}_{S_3} =$_____ $p_{S_4} =$_____ $\hat{p}_{S_5} =$_____

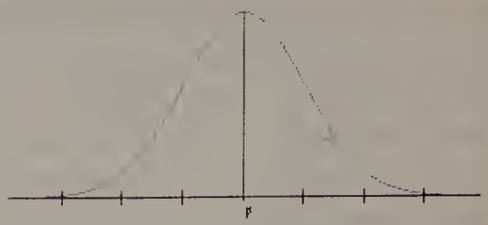
3. Describe the sampling distribution (Center, Shape, and Spread) for the sample proportion of hearts in a sample of 40 cards from a fair deck:
   (*a*) Center, $p =$ _____
   (*b*) Shape _____
   (*c*) Spread, $\sigma_{\hat{p}}$ _____
   (*d*) How do the six sample statistics (five from the calculator and one from Mr. Dempsey's deck) compare with the parameter $p = .25$?

From the February 2013 issue of MATHEMATICS teacher

Activity: Exploring Confidence Intervals

Part 4: Estimating with Confidence

1. Use the normal distribution curve at right to represent the sampling distribution for the proportion of hearts in a fair deck. Draw on the graph approximately where $\hat{p}_D$ and $\hat{p}_S$ fall in relation to the parameter $p = .25$.



2. Of the locations for $\hat{p}_D$ and $\hat{p}_S$, which one is closer to $p = .25$?

3. If we took more samples from the calculator and from Mr. Dempsey's deck, would you expect similar results?

4. Consult with several other groups and compare your results. Did other groups have similar results?

5. Describe any similarities between your results and the other groups' results.

6. Does having more data make you more certain about Mr. Dempsey's guilt or innocence?

7. The empirical rule estimates that 95% of all observations in a normal distribution will fall within two standard deviations of the mean. For a more precise value, we could use 1.96 standard deviations. Knowing this, we can add and subtract 1.96 standard deviations

$$\sqrt{\frac{p(1-p)}{n}}$$

to our point estimate, $\hat{p}$, and associate a 95% level of confidence that such an interval will contain the parameter of interest p. By 95% confidence, we mean that if we were to take many samples, 95% of them would contain the true parameter of interest, p.

Part 5: Constructing the Intervals

1. We will call $\hat{p}_D$ the true proportion of hearts in Mr. Dempsey's deck. If his deck is fair, what is $\hat{p}_D$?

2. Using your sample statistics $\hat{p}_D$ and $\sigma_{\hat{p}}$ compute the following 95% confidence interval for using

$$\hat{p}_D \pm 1.96\sqrt{\frac{\hat{p}_D\left(1-\hat{p}_D\right)}{n}}.$$

3. Does your interval contain the parameter $\hat{p}_D = .25$?

4. Using your five sample statistics for $\hat{p}_S$ and $\sigma_{\hat{p}}$, compute five 95% confidence interval for $\hat{p}_S$ using

$$\hat{p}_S \pm 1.96\sqrt{\frac{\hat{p}_S\left(1-\hat{p}_S\right)}{n}}.$$

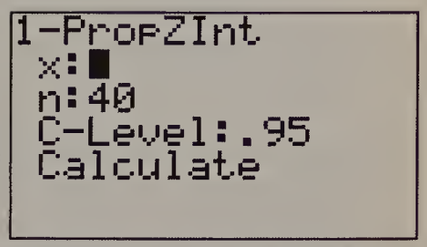
Given the data that you have collected, do you think that Mr. Dempsey is unfair? Explain why or why not.

Activity: Exploring Confidence Intervals

Part 6: Using Technology

The graphing calculator can be used to make our task even easier.

1. Press the **STAT** key and select **TESTS**.
2. From the **TESTS menu**, select **A:1-PropZInt**.
3. Enter the number of hearts that you found in your simulation as x, $n = 40$ trials. The confidence level must be entered as a decimal, so for a 95% confidence level enter **.95**. Select **Calculate** and press **ENTER**.
4. Did the calculator produce the same interval that you computed in part 4?



EXTENSION

1. Would you expect similar or different results if we increased the number of trials? Why or why not?
2. Would you expect different results if the sampling were done without replacement?

Five Fair Decks (from Calculator)			
Hearts	**Clubs**	**Diamonds**	**Spades**
Total =	Total =	Total =	Total =

Hearts	**Clubs**	**Diamonds**	**Spades**
Total =	Total =	Total =	Total =

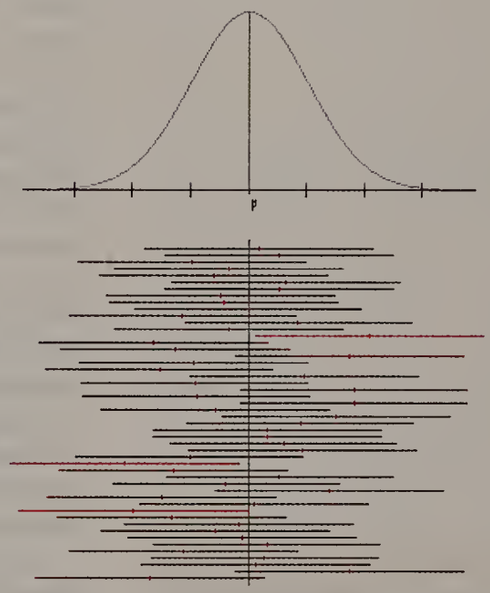
Hearts	**Clubs**	**Diamonds**	**Spades**
Total =	Total =	Total =	Total =

Hearts	**Clubs**	**Diamonds**	**Spades**
Total =	Total =	Total =	Total =

Hearts	**Clubs**	**Diamonds**	**Spades**
Total =	Total =	Total =	Total =

SUMMARY OF CONFIDENCE INTERVALS

The term *95% confidence* means that if you did the simulation 100 times and made 100 intervals, about 95 of them would contain the true value of the parameter $p = .25$.



Observe the results of 50 simulations above drawn from a fair deck. Three intervals do not contain p, because the $\hat{p}$ for those intervals was unusually far from p. The interpretation of the interval is far more important than the computation.

Corey Webel

Classroom Collaboration: Moving beyond Helping

Do you use group work in your mathematics class? What does it look like? What do you expect your students to do when they work together? Have you ever wondered what your students *think* they are supposed to do?

In a recent research project, I investigated the perspectives of eight high school students whose teacher regularly asked them to work collectively to solve open-ended problems. I observed twenty-three classes during the semester and interviewed each of the eight students twice, inquiring about their goals related to working together. During the second interview, I showed each student clips of his or her own participation in small group work and asked each to reflect on the recorded episode (Kotsopoulos 2008).

Although the teacher, Mr. Neal (all names in the article are pseudonyms), described the importance of students collaborating to generate solutions together, not all the students I interviewed described their goals for working together in the same way. Some students emphasized nonmathematical goals, such as working with friends or getting through the tasks, but others emphasized mathematical goals, such as learning multiple strategies. Some students seemed to place responsibility for meeting goals primarily on individuals, emphasizing behaviors such as paying attention and "getting help." A focus on getting and receiving help implies that some students will be treated as "experts" whereas others are treated as "novices" and are helped by the experts. This is sometimes referred to as *asymmetric positioning* (Esmonde 2009). A few students, however, described solving problems as a group responsibility, emphasizing behaviors such as resolving disagreements, collecting ideas, and wanting to be convinced. These behaviors suggest a more *symmetric* positioning among students.

In this article, I focus on three students—Logan, Sevanye, and Connor—to illustrate these contrasting views. Each student took on a unique role while working in groups. However, their actions along with their reflections revealed that Logan and Sevanye had similar perspectives about how collaboration should work.

SIMILAR PERSPECTIVES: LOGAN AND SEVANYE

During one of my class visits toward the end of the school year, Mr. Neal asked his students to work together on the following task (Hirsch et al. 2008):

> Kent County has $200,000 to spend on student salaries.
> (*a*) How many student workers can be hired if the county pays $2,000 per worker for a summer contract covering eight weeks? What if

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the county pays only $1,500 per worker? What if the county pays only $1,000 per worker?

(*b*) If the pay per worker is represented by p, what function $h(p)$ shows how the number of students who could be hired depends on the level of pay offered?

(*c*) Sketch a graph of the function $h(p)$ and write a brief description of the way $h(p)$ changes as p increases.

(Before continuing, try the task yourself and make some conjectures about how students might approach it.)

Mr. Neal introduced the task by asking students about their own experiences with summer jobs and about how much they would like to be paid. Then he instructed the class to complete the task in groups of their own choosing. Mr. Neal's typical practice was to break group work up into short segments during class, usually having students work on only one task per segment (see Webel 2010). On this day, he announced that he was going to try a different strategy—he would give students a whole period to work in groups on an entire investigation, collecting their work at the end.

Sevanye and Lindsay formed a group, which Logan later joined. The following episode ensued. As you read the episode, consider the effectiveness of the interactions involved.

Sevanye takes the calculator from Lindsay and begins to punch buttons. "First, you gotta see … for $2,000 times 8 weeks … Whoa, what am I doing?" She writes something on her paper. "What is that,16,000?" She checks her calculator again. "Yeah, 16,000."

Lindsay, writing and trying to keep up, leans over to look at Sevanye's paper. "Sixteen, hold on …"

"You take $200,000," Sevanye explains, "and divide it by [16,000, and you get] 12 workers." She writes on her paper. Lindsay alternates between looking at Sevanye's paper and writing.

The two girls continue to work in silence for nearly a full minute. Lindsay continues writing and looking at Sevanye's paper but then gets distracted by an off-task conversation in an adjacent group. When she looks back at Sevanye's paper, she frowns and asks, "What is all this math?"

Sevanye explains, "You do $200,000 divided by16,000, 'cause that's how much each worker gets paid. And that equals 12, so you can hire 12 workers …. Two hundred, divide it by the number, 12,000, so then 16 workers. And then you go and divide 2,000 by 8,000."

At this point, Logan pulls up a chair to join Sevanye and Lindsay. "What are we doing, number 1?" he asks.

The group gets sidetracked for the next couple of minutes, discussing college applications. Then Sevanye, looking back at her book, suddenly raises her hand. "Mr. Neal!"

Mr. Neal comes over to the group. "Yes?"

"I don't understand what number 2—or, well, what 1(*b*)—is asking."

While Mr. Neal and Sevanye start to discuss the problem, Logan and Lindsay appear to read problem 1(*b*) for themselves. But as Mr. Neal tries to convince Sevanye that her solution to problem 1(*a*) contains an error, Logan and Lindsay again get distracted and start having another side conversation. So far, the group work session has lasted for about eight minutes.

This episode reflects notable aspects related to interactions between participants. Sevanye appeared to misinterpret the $2,000 as referring to the amount per worker *per week* rather than the amount for the entire 8-week period. After reading part (*b*), Sevanye seemed to be confused and sought help from Mr. Neal. Logan and Lindsay seemed to accept Sevanye's work as correct.

Sevanye's Perspective

After watching the video clip of this episode with Sevanye, I asked her why she never asked Lindsay or Logan to share their ideas about the problem.

Sevanye: They didn't know what they were doing. They were copying my answers.

Interviewer: Right … so did you assume before that they weren't going to be able to help if you asked them?

Sevanye: I assumed it, 'cause I had asked her, like, a little bit, I asked Lindsay a little bit about, like, what she was doing, and she didn't really seem to know. If I [had] asked her some more, she probably would have confused me more than what I was already confused about … so I just went right to Mr. Neal.

Interviewer: So were you not asking them because you thought that they wouldn't be able to help or because they didn't want to help?

Sevanye: Because I didn't think that they would be able to.

I found it interesting that Sevanye did not think that Lindsay or Logan would be able to help; she never actually asked them what they thought about the problem (although in the interview she claimed that she did). She assumed that the only one who could provide assistance was Mr. Neal. In her actions and her explanation, Sevanye was revealing the belief that contributions can be made only by those who already understand the problem. In fact, when initially asked why teachers use group work, she said, "You put the stronger people with some people who may not know the

In her actions and her explanation, Sevanye was revealing the belief that contributions can be made only by those who already understand the problem.

subject as well, and it helps them." In essence, she was saying that working in groups involves asymmetric positioning.

Sevanye also reiterated throughout her interviews that she did not like working in groups, a reasonable view: Because she does much of the work, Sevanye appears to get very little out of group work. In her interactions, she helps others primarily by providing explanations.

Logan's Perspective

In Logan's interview, I asked him about his participation in the episode, focusing on why he was not listening to Sevanye and Mr. Neal's conversation about the summer wages problem.

Logan: [Sevanye] kind of left me and Lindsay out to dry, and we were kind of over there doing our own thing. So we didn't know what was going on.

Interviewer: Would you have preferred it if she had said, "Hey, what do you guys think about this problem?" instead of going straight to Mr. Neal?

Logan: Probably. It would've just been, like, more interesting to me if she tried to involve me. But [she and Mr. Neal] just left us out, like we weren't even part of their conversation.

Interviewer: So you don't feel that it was your choice to remove yourself from the conversation?

Logan: Uh-uh. It just ended up that way because Sevanye did all of the talking.

In this excerpt, Logan describes feeling left out of the mathematical conversation between Sevanye and Mr. Neal. One might argue that he never made an effort to contribute, but, at the same time, it is true that he was never asked (by his group mates or Mr. Neal) what he thought about the problem. When I asked him if he planned to find out later what Sevanye and Mr. Neal were discussing, he admitted, "I was going to copy it." This response suggests that he never planned to press his group to explain or justify their answers.

For Logan, working together seemed to be about getting the answers from someone who already understood the problem. This theme was repeated throughout my interviews with Logan, in which he indicated that he was "not the brightest kid" and that he preferred explanations from Mr. Neal or from the "smartest kid" he could find.

A DIFFERENT PERSPECTIVE: CONNOR

Connor, another student in Mr. Neal's class, had a different perspective on working in groups. The following episode took place earlier in the school year, in an introductory lesson on functions. Mr. Neal had used two segments of group work already in the ninety-minute period, and this episode occurred about fifteen minutes before the end of the class.

"'Explain why light is a function of distance from light source to a receiving surface,'" Seth reads from the textbook. He pauses. "The more intense the light, then the more light...."

Connor speaks up: "Because it depends on how far ... receiving source is away from it ... until the [*speech trails off*]. Light intensity ... I don't know what that is. I'll have to think about it."

Seth pauses and then says, "If it's closer ... it'll be more intense."

"Yeah," Connor agrees. "Okay. So it would be like, the light intensity is a function of distance from light source. The farther away the receiving source is, the lower intensity it's going to be. And how close it is, the higher intensity."

"Cool." Both students write in their notebooks.

In this episode, both students make contributions to solving the problem. Connor does not function as an expert or a novice. He is willing to offer his ideas even when he is not confident about them ("I don't know what [light intensity] is. I'll have to think about it."). After watching this clip, I asked Connor to talk about this phrase.

Connor: Yeah, when I was doing that, like, I didn't really understand. So I was just kind of thinking out loud, and if Seth had his opinion, he could tell me, like, what he thinks, whether that's right or wrong. I was basically throwing ideas out there. I think it's good to share your ideas and what you're thinking. You don't really have to figure it out for yourself first.

This episode and Connor's explanation reveal that solving problems seems to be, for Connor, a task that involves contributions from several people. Ideas need not be "correct" or accompanied by confidence. "Thinking out loud" is an intermediate step that provides an opportunity for Seth to make a contribution, moving the group forward.

At times, Connor did seek help from Mr. Neal, but he thought of this as a last resort:

Connor: But if we do need help, we will go to Mr. Neal, but we try to explain it to each other as much as we can so that we really understand ... If we all understand it, then we move on the next question.

Here Connor emphasizes that solving problems is a job for the group; everyone needs to understand before they move on to another problem. Group work is about members of the group making contributions (perhaps before knowing whether they are correct) and coming to agreement through discussion. This perspective contrasts with those of Logan and Sevanye, who seem to view group work as an opportunity for knowledge to be delivered from one participant to another in an asymmetric relationship.

Solving problems seems to be, for Connor, a task that involves contributions from several people.

IMPLICATIONS

Interviewing these students allowed me to see that even though Logan and Sevanye participated differently, their views about what it means to work together were quite similar. In contrast with Connor, they both seemed to see working together primarily in terms of getting or giving help, where students are positioned asymmetrically. Sevanye found herself assuming the role of explainer, whereas Logan was relegated to a novice role, dependent on experts like Sevanye or Mr. Neal.

The view of working together as helping is problematic with respect to productive group work. In situations in which no group member can function as an expert, students with this view have no choice but to call on the teacher, as we saw Sevanye do in this episode. They will miss simple errors, as her group did, because Lindsay and Logan were not thinking critically about what Sevanye was doing. They will not persist in solving difficult problems but, instead, will seek a knowledgeable person to tell them how to solve the problems. Students like Connor, on the other hand, will persist in the face of challenge because they believe that even if no one in the group knows immediately what to do, sharing ideas can lead to progress.

How do we help students see that working together is something more than just helping and getting help? One thing we should probably *not* do is explicitly or implicitly endorse asymmetric positioning. For example, Jansen (2012) describes how one middle school teacher assured students that he had placed a "genius" in each group to help them if they were stuck. This strategy (and explaining the strategy to students) appeared to encourage the students to seek out expert peers who could act as mathematical authorities and generally led to less productive mathematical discussions. This finding also raises questions about the potential impact of such strategies on students' sense of their own mathematical abilities.

Mr. Neal did believe that students could work together without relying on an external authority. In an interview, he said:

> ***Mr. Neal:*** When we do work in groups, the goal is to have as many thoughts and opinions and ideas as possible so that together, collectively, that group or the whole class can determine what it is that is correct … as opposed to me saying this is right, this is not.

But in practice, Mr. Neal's instruction may have at times unintentionally encouraged asymmetric positioning.

With regard to the episode involving Logan and Sevanye, Mr. Neal might have been able to promote a more symmetrical approach by avoiding an extensive dialogue with Sevanye. Such one-on-one dialogue reinforces the help-giving model (see Webb, Nemer, and Ing 2006) and releases Logan and Lindsay from any responsibility for making contributions.

Instead, Mr. Neal could have elicited ideas from Logan and Lindsay, a strategy sometimes referred to as *social scaffolding* (Dekker and Elshout-Mohr 2004). For example, he could have addressed Logan and Lindsay by asking, "Have you shared your ideas about this problem?" and addressed the entire group by asking, "Did the group come to an agreement on the first part of the problem?" By doing so, he would indicate the value of each person's contributions to the group.

Mr. Neal might have then praised Logan's and Lindsay's contributions, a practice that affirms student competence and encourages future participation (Boaler and Staples 2008). Mr. Neal might also have drawn on the class community by stopping and having multiple groups share different ideas, a strategy that could remove him from the help-giving role and shift authority to the class and to the mathematics (Webel 2010).

All these strategies emphasize generating, testing, comparing, and revising ideas rather than getting ideas from experts.

WORKING TOGETHER

Both Logan and Sevanye adopted roles that were reasonably connected to their expressed view that working together is mostly about helping or getting help. This view should encourage us to create opportunities for students to experience and see working together as a collaborative endeavor, as Connor did, in which the goal is to build knowledge collectively by listening to and reflecting on one another's ideas.

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Maria L. Hernandez and Nils Ahbel

Using FluidMath to Explore Recursive and Explicit Functions

Technology Tips, which provides a forum for innovative uses of technology in the teaching and learning of mathematics, appears seven times each year in *Mathematics Teacher*. Manuscripts for the department should be submitted via http://mt.msubmit.net. For more background information on the department and guidelines for submission, visit http://www.nctm.org/publications/content.aspx?id=10440#tech.

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FluidMath™ (www.fluiditysoftware.com), a new mathematics software tool for Tablet devices, computers, and interactive whiteboards, can create a dynamic graph or table with a simple gesture and recognize written expressions as the mathematical relationship they intend. The software uses a stylus as its input device. By changing constant values in an equation to parameters, the user can create sliders instantly and see graphs and tables change dynamically. The CAS (Computer Algebra System) functionality allows simplification of algebraic expressions and solution of equations and can perform all the calculations from algebra through calculus. FluidMath uses standard mathematical notation to explore explicitly and implicitly defined functions, parametric functions, polar functions, and recursively defined functions.

Here we will first use this software to help students investigate recursive equations, which are used as a tool to define a process that we want to carry out over and over again.

Consider the following problem:

> Suppose that Andrew has a headache and takes 400 mg of ibuprofen. His body eliminates 67% of the drug every 4 hours.

We write a set of recursive equations that represents the amount of the ibuprofen in Andrew's system after several 4-hour time periods. We let $D(0)$ represent the initial amount of ibuprofen in Andrew's system and $D(n)$ represent the amount of ibuprofen in his system after n 4-hour periods. The amount of drug in his system at any 4-hour period is equal to the previous amount of drug minus what the body eliminates in that 4-hour period, giving the following equations:

$$D(0) = 400 \text{ and}$$
$$D(n) = D(n-1) - 0.67D(n-1)$$

As students write the recursive system of equations, FluidMath "recognizes" their handwriting and produces the mathematics in standard print form just below the handwriting (see **fig. 1**).

Because we have defined an expression recursively, the software can now

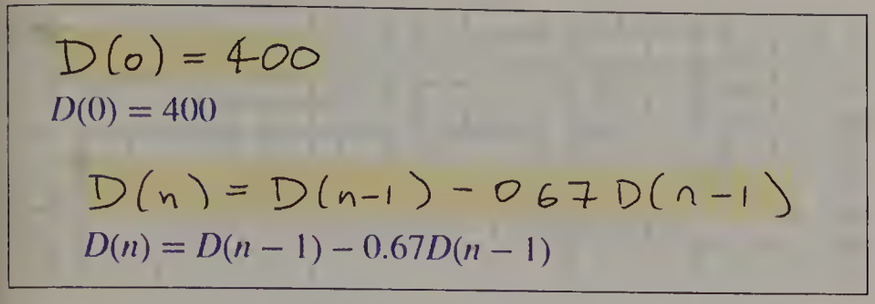

Fig. 1 FluidMath will automatically convert handwritten equations to print, including the appropriate use of italics for variables.

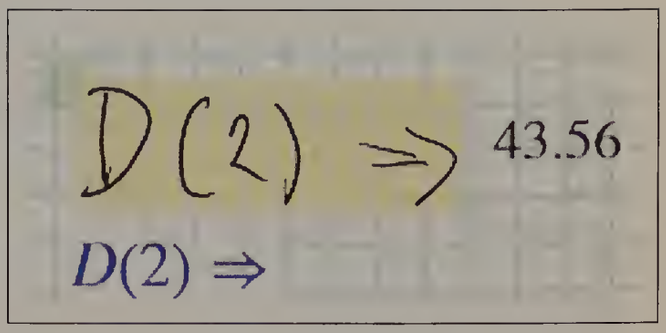

Fig. 2 FluidMath will evaluate the recursive function at $n = 2$.

evaluate the function for a particular 4-hour period. For example, we can calculate the amount of drug in Andrew's system after 8 hours, or two 4-hour periods (see **fig. 2**). The double-arrow symbol indicates that we want the approximate value of the expression that precedes it. In this case, we can see that the amount of drug in Andrew's system is around 43.56 mg.

We can also produce a graph of the discrete values produced by the recursive set of equations. To do so, we make what is known as a "graphing gesture" through the set of equations (see **fig. 3**).

To adjust the graphing window, simply tap on the top arrow on the y-axis and write in a value such as **500** to change the maximum value. Hovering over the origin until crosshairs appear and dragging diagonally away from the origin will also change the graphing window dynamically (see **fig. 4**).

Clicking on the checkbox by the **Table** tab at the top of the graphing screen will produce a list of numerical values for the recursive function. As shown in **figure 5**, we can display all three representations for this problem simultaneously: algebraic, graphical, and numerical.

The next step in considering this set of recursive equations might be to write a closed-form, or explicit, function representing the problem. The **Draw** feature of FluidMath allows us to write in the file without having the software interpret the writing mathematically, almost like a **comment** feature. Colors available in the **Draw** menu box at the top left of the screen allow teachers to communicate the development of the explicit function more effectively. **Figures 6** and **7**, respectively, show the **Draw** menu and

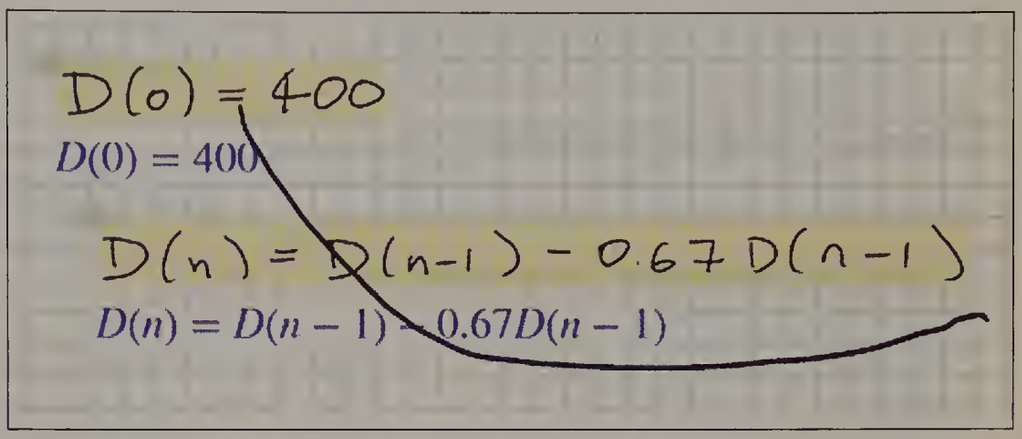

Fig. 3 Sketching an arc through both equations will produce a graph of the equations.

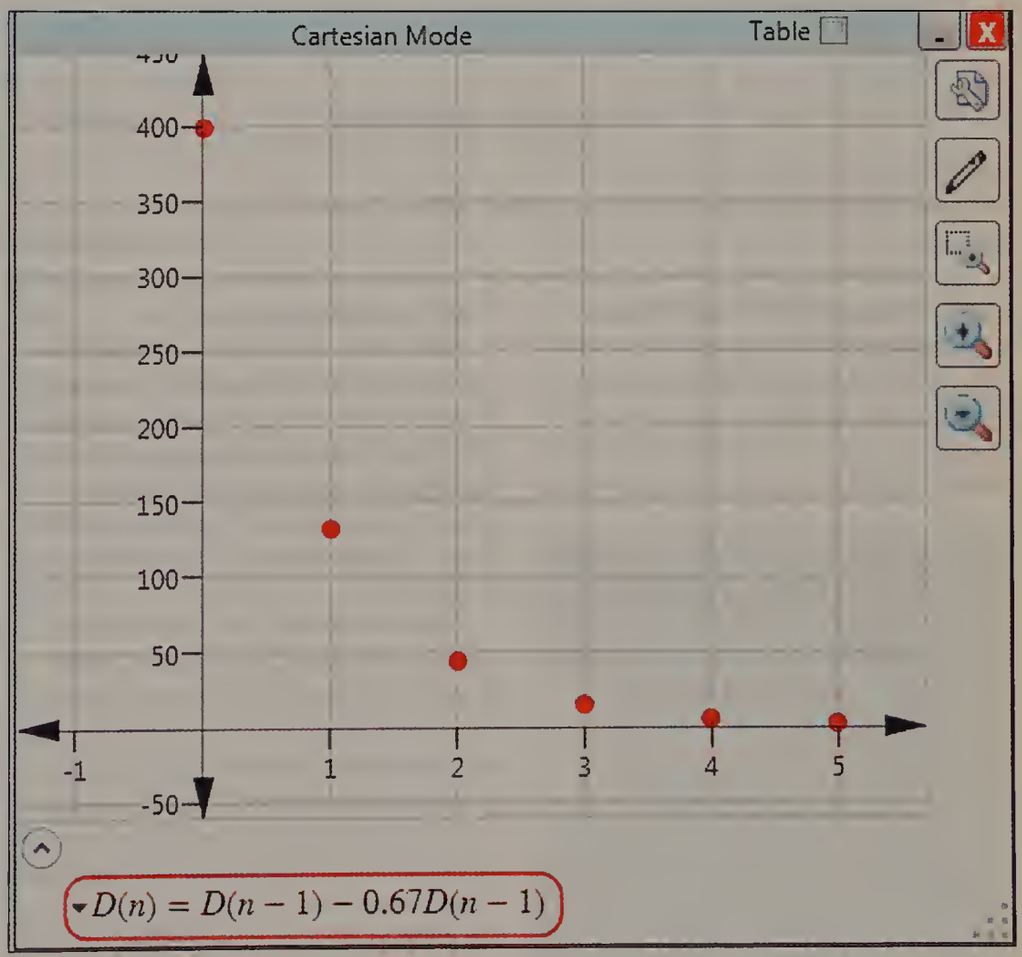

Fig. 4 The gesture depicted in **figure 3** produces a graph of the discrete values for the amount of drug in Andrew's system for the first five 4-hour periods.

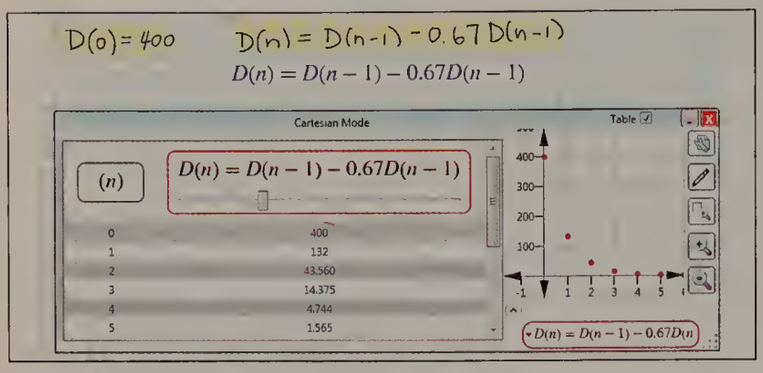

Fig. 5 A display of all three representations for this problem—algebraic, graphical, and numerical—can be shown on the same screen.

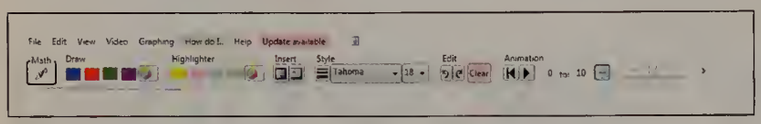
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Fig. 6 The **Draw** menu can be used to annotate the document.

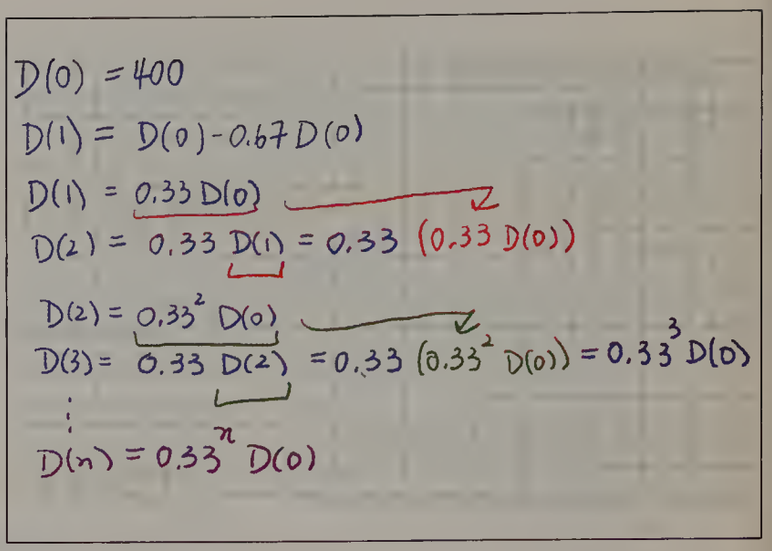

Fig. 7 We can create a closed-form function for the amount of drug in Andrew's system.

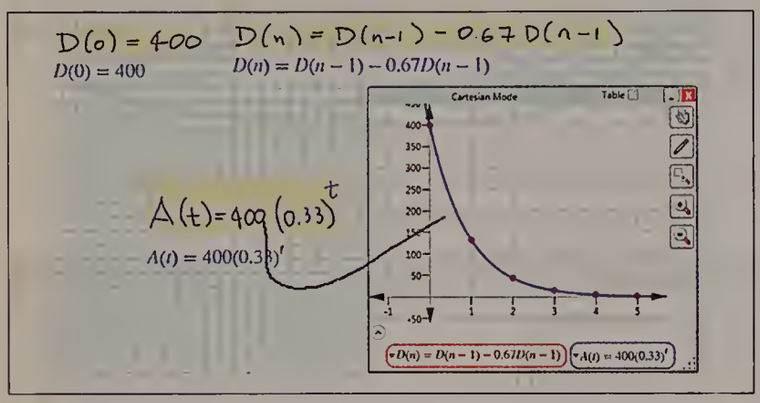

Fig. 8 Users can produce a graph of the continuous closed-form function along with the graph of the discrete values generated by the recursive functions.

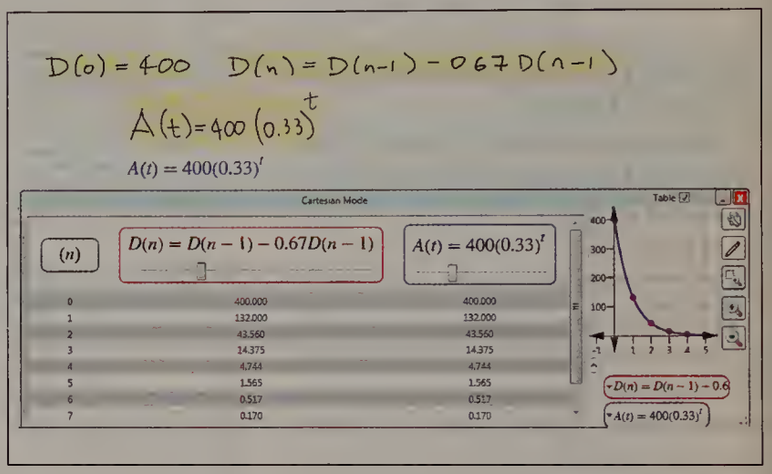

Fig. 9 Students can compare the values generated by each function.

the development of the explicit form.

Using the **Draw** menu and our recursive equations, we can write out some equations for specific values of n and develop the closed-form function for the amount of drug in Andrew's system (see **fig. 7**).

So now we have an explicit function, $A(t)$, which represents the amount of drug in Andrew's system after t hours. Because this function has now become continuous rather than discrete, we change the variable to t for time. If the initial dose is 400 mg, then $A(t) = 400(0.33)^t$.

We can define this function and graph it along with the discrete values (see **fig. 8**). To add the graph of the continuous function to the graph of discrete values, we make a graphing gesture that starts in the definition box for $A(t)$ and ends in the graphing window.

The numerical values for the different forms can be compared by using the Table feature. As before, click on the checkbox to display the values in a table. In **figure 9**, we can see that the discrete values generated by the recursive function match those of the continuous function.

Students can see from this example that FluidMath offers a way to demonstrate the connection between the discrete-valued, recursive equations and the closed-form, continuous function. That is, for integer values of t, the values of $A(t)$ match those of the recursive function $D(n)$. Students will make the connection that repeated multiplication by the same factor creates a geometric sequence of values that can be modeled by an exponential function.

USING FLUIDMATH TO TRANSFORM FUNCTIONS

The software's dynamic interface can also be used to help students understand transformations of functions. Consider the general function $f(x) = p \sin \cdot (q(x + r)) + s$. We can explore the effects of each of the parameters p, q, r, and s on the graph of the parent function $y = \sin x$ (see **fig. 10**).

Let's define the function f as shown in **figure 10** and set the values of r, s and p. The default value for each of these parameters is 1. Then we create a graph of f using the graphing gesture and add the graph of $y = \sin x$ to the existing graph.

Now we can create a slider for q by using the slider gesture. This is a vertical mark that begins on the equals sign and ends outside the definition box for q.

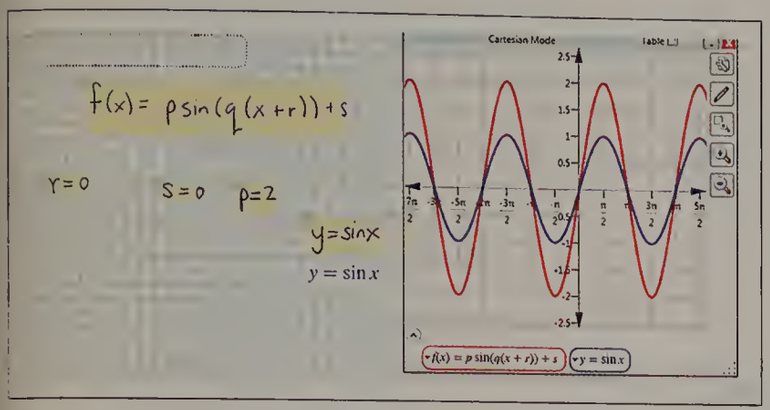

Fig. 10 Students can use parameters in a function definition, define values for those parameters, and graph the function along with the "parent" function.

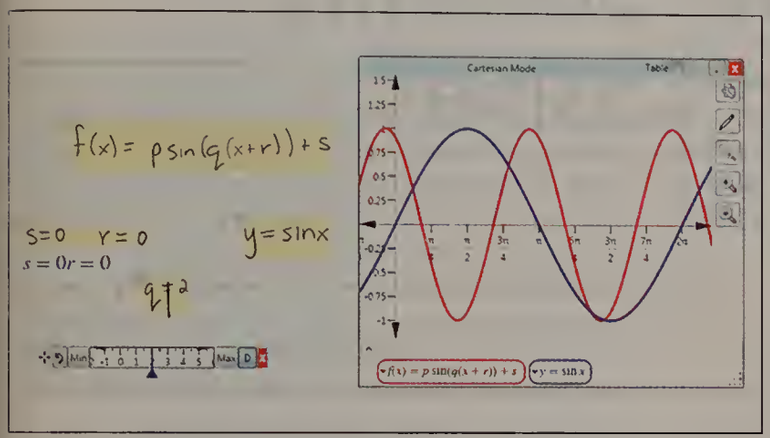

Fig. 11 Creating a slider can dynamically change the values of each parameter.

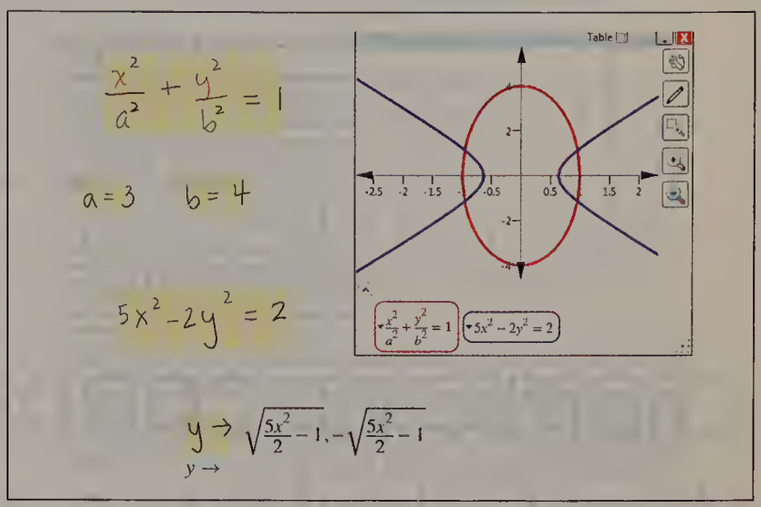

Fig. 12 FluidMath will solve an implicit equation for one variable in terms of another.

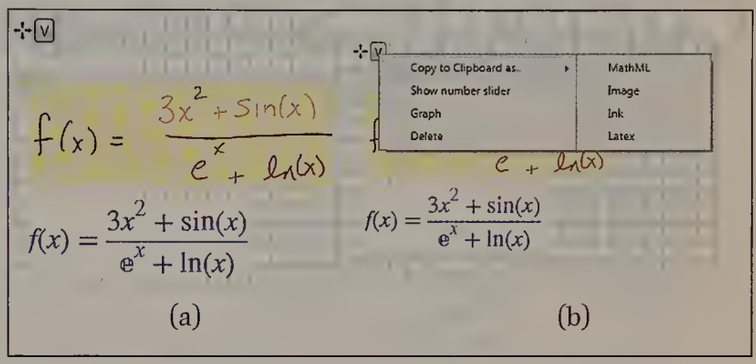

Fig. 13 FluidMath creates typeset mathematical expressions (**a**) for import into other applications (**b**).

Dragging the blue triangular widget on the slider will change the graph dynamically (see **fig. 11**).

FluidMath also graphs implicit, polar, and parametric functions. **Figure 12** shows a few examples of these capabilities.

Another advantage of using FluidMath is the ability to create mathematical equations in standard print format. After handwriting an expression in FluidMath, we can hover over the dot in the top left corner of the expression, as shown in **figure 13**. Click on **Copy to Clipboard as...** and then choose **MathML**. We can easily paste the typeset mathematics expression into a Word document. This feature is a timesaver when creating examples for a handout or problems for assessments.

FLUIDMATH AND THE COMMON CORE STANDARDS

As these examples show, FluidMath is a powerful tool that can help teachers and students explore the behavior of functions and make sense of mathematical concepts across a variety of topics. The Common Core State Standards for Mathematics (CCSSI 2010) states that students should be able to use appropriate tools strategically as follows: "When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data.... They are able to use technological tools to explore and deepen their understanding of concepts" (p. 7). FluidMath is an example of how students and teachers can get right to the heart of mathematical discovery without the burden of learning complicated commands or accessing multiple menus.

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Bryan C. Dorner

Chordic vs. CORDIC: How Calculators and Students Compute Sines and Cosines

Students who have grown up with computers and calculators may take these tools' capabilities for granted, but I find something magical about entering arbitrary values and computing transcendental functions such as the sine and cosine with the press of a button. Although the calculator operates mysteriously, students generally trust technology implicitly. However, beginning trigonometry students can compute the sine and cosine of any angle to any desired degree of precision using only simple geometry and a calculator with a square root key.

Delving Deeper offers a forum for classroom teachers to share the mathematics from their own work with the journal's readership; it appears in every issue of *Mathematics Teacher*. Manuscripts for the department should be submitted via http://mt.msubmit.net. For more background information on the department and guidelines for submitting a manuscript, visit http://www.nctm.org/publications/content.aspx?id=10440#delving.

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This article describes two algorithms, chordic and CORDIC, intended to remove the mystery of how sine and cosine functions can be computed. Each algorithm uses the familiar setting of the unit circle. Many calculators use the CORDIC algorithm to compute sines and cosines. I call the algorithm that I use in my classes the "chordic algorithm," because it uses chords on the unit circle. In this article, I show how each of these algorithms can be explained using elementary analytic geometry. These explanations could be transformed into guided discovery lessons within the reach of beginning trigonometry students.

When students use the chordic algorithm, they reason about radii and chords of circles and connect them to trigonometric ratios, the setting in which trigonometry arose historically. The pedagogical advantages of such appeals to the original context from which the ideas arose have been nicely described in articles by Bressoud (2010) and Suzuki (2009).

THE CHORDIC ALGORITHM

We begin by finding the sine and cosine of an angle halfway between two angles whose sine and cosine are already known (see **fig. 1**). Once this method is understood, we use it repeatedly to approximate the sine and cosine of any angle.

The three-step algorithm finds the sine and cosine of an angle x that is located exactly halfway

between two angles, $x + h$ and $x - h$, whose sine and cosine values are previously known. For example, it will find the sine and cosine of 45°, which is halfway between 0° and 90°, where we know that $\sin 0° = 0$, $\cos 0° = 1$, $\sin 90° = 1$, and $\cos 90° = 0$.

Let P and Q be points with coordinates $(\cos(x - h), \sin(x - h))$ and $(\cos(x + h), \sin(x + h))$ on the unit circle corresponding to the angles $(x - h)$ and $(x + h)$, whose sine and cosine values are known.

The three-step chordic process follows:

1. Find the coordinates of point A with coordinates (a, b) midway between P and Q where

$$a = \frac{1}{2}(\cos(x + h) + \cos(x - h))$$

and

$$b = \frac{1}{2}(\sin(x + h) + \sin(x - h)).$$

(Note that OA bisects angle POQ. Hence, A and B lie on the line at angle x, halfway between the angles whose sine and cosine are known.)

2. Find the distance $OA = \sqrt{a^2 + b^2}$. (Note that $OA = \cos h$.)
3. Use right triangle OAC to compute the sine and cosine of x and thus find the coordinates of point B:

$$\sin x = b/OA = b/\cos h$$
$$\cos x = a/OA = a/\cos h$$

Finding Arbitrary Values between Known Values

Consider the problem of finding the sine and cosine of an arbitrary angle. For simplicity, we can use reference angles to reduce the problem to finding the sine and cosine of an angle, y, in quadrant I. All such angles have measures between 0° and 90°. Knowing the sine and cosine of 0° and 90°, we can use the three-step chordic method to find the sine and cosine of 45°, the angle halfway between 0° and 90°. If we assume that y does not equal 45°, it will lie either between 0° and 45° or between 45° and 90°. Again, use the three-step method to find the sine and cosine of the angle halfway between the two angles bracketing y. Continue in this fashion, finding $\sin x$ and $\cos x$ where, at each stage, x is the angle midway between the angles $x + h$ and $x - h$ that bracket y and whose sines and cosines have already been found. This method is similar to and foreshadows the "midpoint method" for finding the zeros of a continuous function that students may encounter in a later course.

Students can use a spreadsheet to perform the algorithm one iteration at a time. **Figure 2** shows the initial values and the formulas used in an Excel® spreadsheet.

To iterate the procedure, copy the values for **Middle Angle** and its sine and cosine and paste them as values in place of either the values for **Big Angle** or **Small Angle** so as to keep the desired angle, y, bracketed. Take care to select **Paste Special** and then **Values**; otherwise, Excel will try to paste the formulas, resulting in an error.

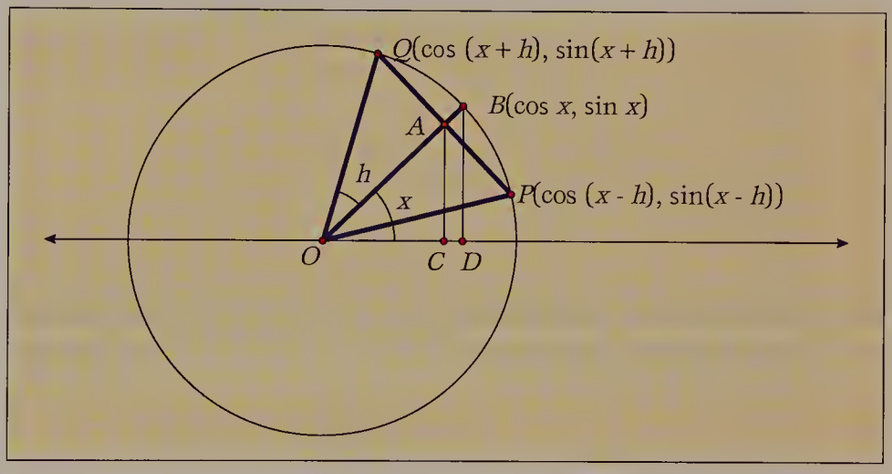

Fig. 1 The chordic algorithm finds the sine and cosine of an angle halfway between two angles whose sines and cosines are known.

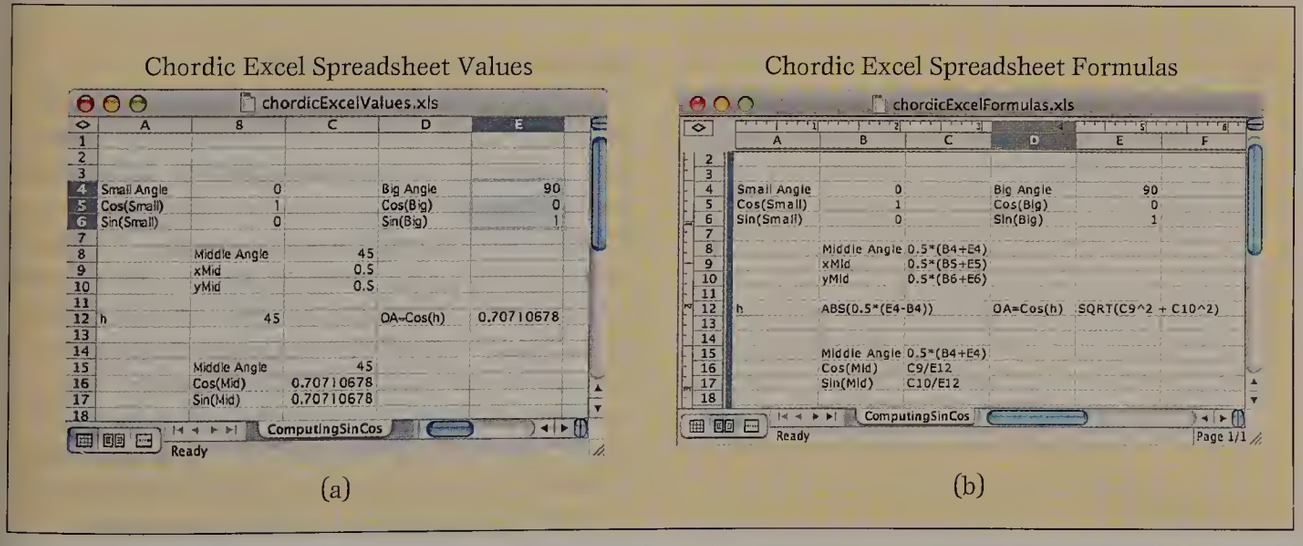

Fig. 2 The computation is made easier when a spreadsheet is used.

The approximations converge quite rapidly. To see this, suppose that P and Q are points on the unit circle that correspond to angles whose sine and cosine values are known (see **fig. 3**) and that are being used to compute the sine and cosine of the angle x. Assume that the desired angle is y. In **figure 3**, we see that

$$|\cos y - \cos x| < RP,\ RP < PQ, \text{ and } PQ < \overset{\frown}{PQ}. \quad (1)$$

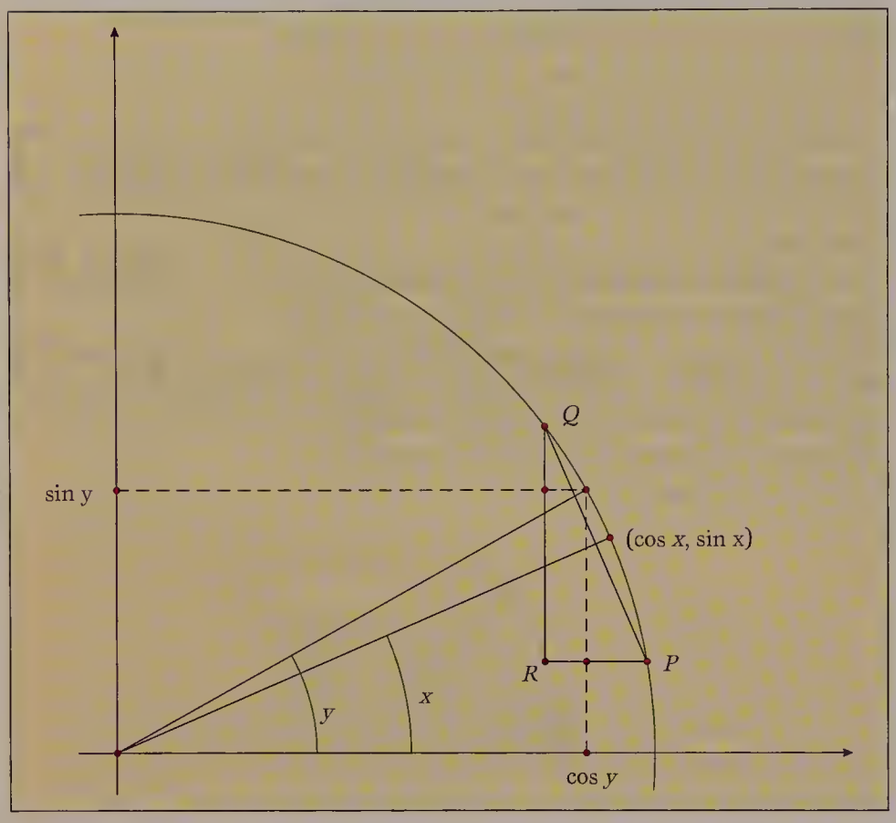

Fig. 3 Approximate values very quickly converge for the desired angle, y. The error in the kth approximation is less than $\pi/2^k$.

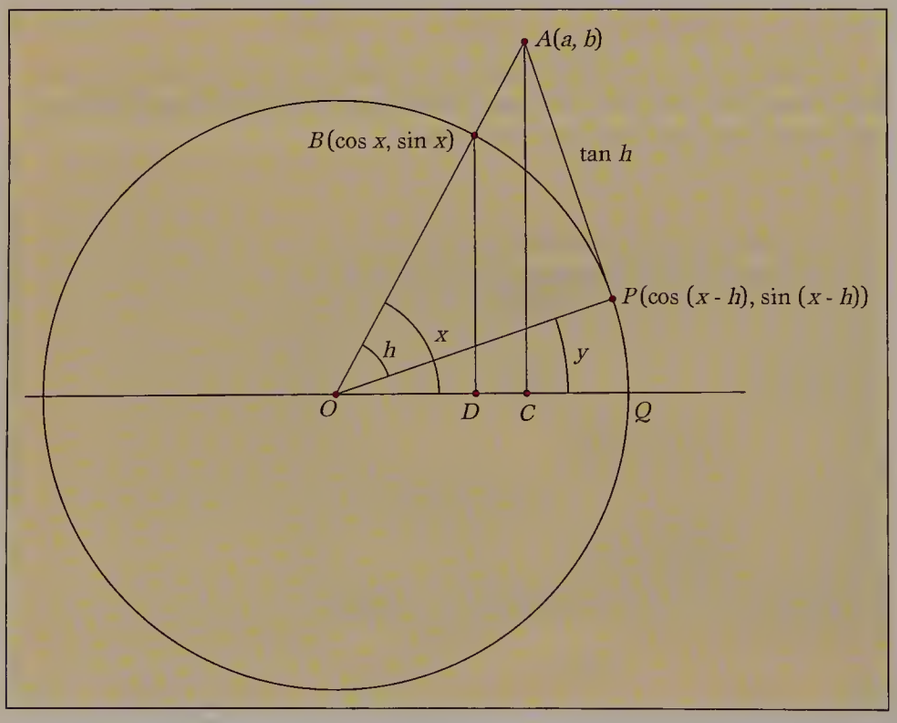

Fig. 4 The tangential process requires determining the coordinates of point A.

Thus, the error in the approximations to the cosine at any stage is less than the difference in radian measure of the angles determined at the previous stage. Similarly, we also see that $|\sin y - \sin x| < PQ$.

By this argument, the error in the first approximation to both sine and cosine is less than $\pi/2$, and at each stage the iteration cuts the error in half. The error in the kth approximation is less than $\pi/2^k$. After a dozen iterations, we can be sure that the error in both the sine and cosine is less than

$$\frac{\pi}{2^{12}} = \left(\frac{\pi}{2^2}\right)\left(\frac{1}{2^{10}}\right) < 1 \cdot \frac{1}{1000} = 0.001. \quad (2)$$

Improving the Chordic Algorithm

Steps 2 and 3 of the chordic algorithm take a great deal of time because the time needed to compute square roots and divide are both significantly greater than the time needed for addition or even multiplication. Having restricted ourselves to angles in quadrant I, we know in advance that the sequence of angles, h, will be $\pi/4$, $\pi/8$, ... Therefore, we can precompute the values of $1/\cos h$ in step 3 of each stage.

Thus, each iteration requires finding only two averages (step 1) followed by two multiplications by $1/\cos h$ (step 3), avoiding both the square root and the division operations in steps 2 and 3. Further, division by 2 when computing the average does not take a computer much time. One simply shifts the "binary" point, just as we move the decimal point when dividing by 10. For example, to find the average of 1 and 4 in decimal notation, we first add $1 + 4 = 5$ and then divide by 2 to get 2.5. Some work would be involved if the last step were carried out in detail. In binary notation, we add $1_{\text{two}} + 100_{\text{two}} = 101_{\text{two}}$ and then just shift the binary point to get 10.1_{two}. Stated generally, when multiplying a number by an integer power of 2, the computer simply has to shift the binary point, an operation that takes it very little time.

THE TANGENTIAL METHOD

After discovering the chordic algorithm, I learned that many calculators use the CORDIC algorithm. Had my work been a rediscovery of the CORDIC method? No. When I managed to derive the CORDIC method using analytic geometry similar to the chordic algorithm, my derivation turned out to be based on tangents rather than chords. Thus, CORDIC is not chordic; it is tangential. In fact, it is a special case of a wider class of tangential algorithms. I will first describe the general tangential algorithm and then refine it to produce the CORDIC algorithm. As in the chordic algorithm, we will restrict ourselves to finding sines and cosines of quadrant I angles.

The general tangential algorithm is also a three-step method, which uses the diagram shown in **figure 4**. As in the chordic algorithm, we seek the sine and cosine of an angle x starting from angles $x \pm h$ where the sine and cosine values of the angles are known. In the chordic case, we needed to know the sine and cosine at both $x + h$ and $x - h$. In the tangential case, we need to know the sine and cosine only at either $x + h$ or $x - h$.

In both the chordic and tangential algorithms, observe the key point: First, we find a point A for which we know both the coordinates (a, b) and also the angle x, the terminal side of which passes through A. Then we use steps 2 and 3 to compute the sine and cosine of x.

In the chordic algorithm, A is chosen to be the midpoint of the chord between the previously known values because we can readily find the angle of the line through A and also the coordinates of A. In the tangential algorithm, we can find the coordinates of A along the tangent through P.

This three-step tangential algorithm finds the sine and cosine of an angle x that is located at an angle, $x = y \pm h$, where y is an angle whose sine and cosine values are previously known.

Let P be a point on the unit circle at the angle $y = x \pm h$ whose sine and cosine are known. P has coordinates $(\cos(x \pm h), \sin(x \pm h))$. **Figure 4** shows the case for the angle $y = x - h$.

The three-step tangential process follows:

1. Find the coordinates of $A = (a, b)$ located a distance $\tan h$ from P along the tangent line to the unit circle at P so that A is on the terminal side of angle x.
2. Find the distance $OA = \sqrt{a^2 + b^2}$. (Note: $1/OA = \cos h$.)
3. Use right triangle OAC to compute the sine and cosine of x and thus find the coordinates of point B:

$$\sin x = b/OA = b\cos h$$
$$\cos x = a/OA = a\cos h$$

Finding Coordinates along the Tangent Line

Step 1 in the tangential algorithm is the crucial step in which we compute the coordinates of point A. To describe this step, it is helpful to consider a more general case (see **fig. 5**): We find the coordinates of a point on the tangent of a general circle of any radius, r, centered at the origin.

Let P be the point with coordinates $(r\cos(x - h), r\sin(x - h)) = (p, q)$ on the circle of radius r centered at the origin, corresponding to the angle $x - h$ whose sine and cosine are known.

We now want to find the coordinates of point $A = (p', q')$ located a distance $r\tan h$ from P along the tangent line to the circle at P so that A is on the terminal side of the angle x. Note that the distance from A to the origin is $r' = r\sec h$.

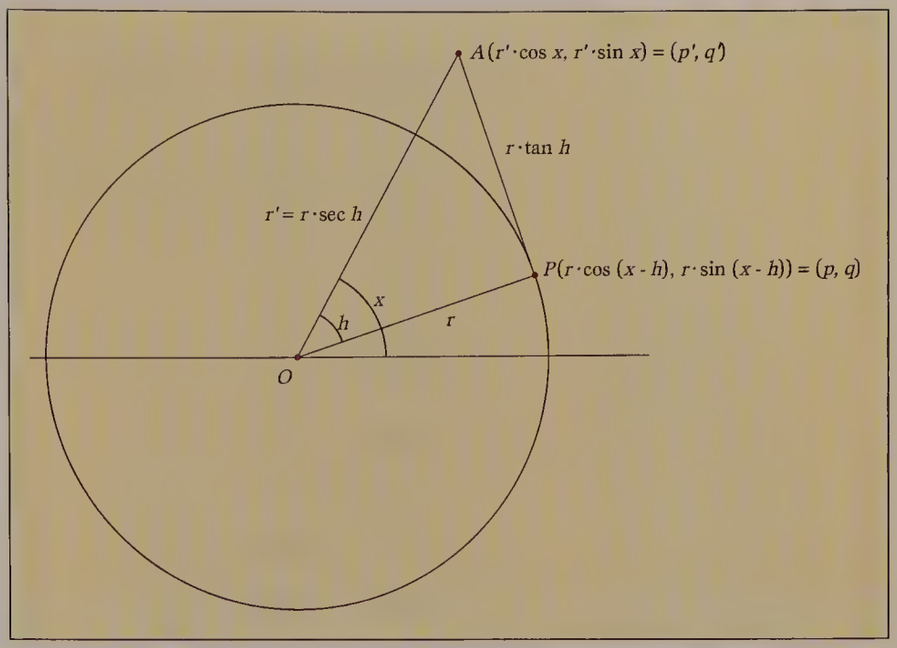

Fig. 5 Considering a more general case is helpful.

The tangent passes through P and is perpendicular to OP. The slope of OP is q/p, so the tangent must have slope $-p/q$. A parametric form of the equation of this line follows:

$$T(t) = (x(t), y(t)) \text{ where}$$
$$x(t) = p - qt$$
$$y(t) = q + pt. \qquad (3)$$

Let's check that this result is correct. The line passes through $P = (p, q)$ when $t = 0$ and has the desired slope:

$$\frac{y(1) - y(0)}{x(1) - x(0)} = \frac{(q + p) - q}{(p - q) - p} = -\frac{p}{q}$$

Point A in **figure 5** is on the tangent line so its coordinates are given by (3) for some value of t. It turns out that the distance from A to P is proportional to t because, according to the distance formula, we have the following:

$$\sqrt{((p - qt) - p)^2 + ((q + pt) - q)^2}$$
$$= \sqrt{(-qt)^2 + (pt)^2} = |t|\sqrt{q^2 + p^2} = r|t| \qquad (4)$$

Note that the proportionality constant is $r = \sqrt{q^2 + p^2}$, the length of the radius, OP. Hence, if we substitute $t = \tan h$ into $T(t)$, we see from (4) and **figure 5** that we have moved through the angle h from P to A, as desired.

According to equation (4), the function $d(t) = rt$ represents the displacement, or signed distance, of A from P. We see from equation (3) that if t is

stage	P angle	P coords	A angle	A coords	A sin	A cos
0	0.000	0.637 0.000	0.785	0.637 0.637	0.707	0.707
1	0.785	0.637 0.637	1.178	0.373 0.900	0.924	0.383
2	1.178	0.373 0.900	0.982	0.552 0.826	0.831	0.556
3	0.982	0.552 0.826	0.884	0.633 0.772	0.773	0.634
4	0.884	0.633 0.772	0.834	0.671 0.741	0.741	0.672
5	0.834	0.671 0.741	0.810	0.689 0.724	0.724	0.690
6	0.810	0.689 0.724	0.798	0.698 0.716	0.716	0.698
7	0.798	0.698 0.716	0.804	0.694 0.720	0.720	0.694
8	0.804	0.694 0.720	0.801	0.696 0.718	0.718	0.696
9	0.801	0.696 0.718	0.799	0.697 0.717	0.717	0.697
10	0.799	0.697 0.717	0.800	0.697 0.717	0.717	0.697

Fig. 6 This sample tangential approximation of the sine and cosine of an angle of 0.8 radians arrives at the respective results of .717 and .697 after ten steps.

step	stage	angle	P coordinates p	P coordinates q	t	arctan(t)	sine	cosine
0	0	0.000	1.000	0.000	1.000	0.785	0.000	1.000
1	0	0.785	1.000	1.000	1.000	0.785	0.707	0.707
2	1	1.571	0.000	2.000	-0.500	-0.464	1.000	0.000
3	1	1.107	1.000	2.000	-0.500	-0.464	0.894	0.447
4	2	0.644	2.000	1.500	0.250	0.245	0.600	0.800
5	3	0.888	1.625	2.000	-0.125	-0.124	0.776	0.631
6	4	0.764	1.875	1.797	0.063	0.062	0.692	0.722
7	5	0.827	1.763	1.914	-0.031	-0.031	0.736	0.677
8	6	0.795	1.823	1.859	0.016	0.016	0.714	0.700
9	7	0.811	1.793	1.887	-0.008	-0.008	0.725	0.689
10	7	0.803	1.808	1.873	-0.008	-0.008	0.720	0.694
11	8	0.795	1.823	1.859	0.004	0.004	0.714	0.700
12	8	0.799	1.816	1.866	0.004	0.004	0.717	0.697
13	9	0.803	1.808	1.874	-0.002	-0.002	0.720	0.694
14	9	0.801	1.812	1.870	-0.002	-0.002	0.718	0.696
15	10	0.799	1.816	1.866	0.001	0.001	0.717	0.697

Fig. 7 This sample CORDIC computation corresponds to that in **figure 6**.

positive, A is located above and to the left of P, as in the case shown in **figure 5**. If t is negative, A is located below and to the right of P. We further note that when $t = \tan h$, the signs of t and h are the same. Hence, informally, for any point $P = (p, q)$ on a circle of radius r, substituting $t = \tan h$ in the parametric equations (3) moves from P along the tangent line and through the directed angle h about the origin. (I was pleased to see this result in the calculation of $d(t)$; I was not previously aware of it.)

The tangential method provides more flexibility than the chordic in two ways:

1. In **figure 1**, the angle h must be exactly half the angle between the radii that bracket the desired angle. However, in **figure 5**, we can find the coordinates corresponding to any angle h where $-\pi/2 < h < \pi/2$.
2. In the chordic algorithm, step 3 is unavoidable at each stage because step 1 of the following stage requires that both points be on the same circle so that a chord may be drawn. The tangential algorithm requires repeating the generalized version of only step 1. Steps 2 and 3 can be omitted until the approximating angle is as close as we wish to the desired angle and then can be carried out just once to find the sine and cosine of our approximation.

If we start from the angle $y = x - h = 0$ and use the values $h = \pi/4, \pm\pi/8, \pm\pi/16, \ldots$, the tangential method will mirror the chordic method, moving through the same angles. In this case, as shown in inequality 2, twelve steps will ensure that the approximating angle will be within 0.001 of the true radian measure.

We should also note in **figure 5** that the distance of A to the origin is obtained by multiplying the distance, r, of P to the origin by $\sec h$ whether h is positive or negative. Hence, if we start at point $P = (1, 0)$ where $r = 1$, after twelve steps we will be at a radius of $r' = 1 \sec h_1 \cdot \sec h_2 \cdot \sec h_3 \cdot \cdots \cdot \sec h_{12}$, independent of the angle that we are approximating. We can precompute this constant and divide by it just once. Even better, we can also predivide and start at $(1/r', 0)$ where $r = 1/r'$. Then twelve steps will bring us to a point where $r = 1$ and the coordinates of our final point will be our final approximations of the sine and cosine without dividing during the computation at all. **Figure 6** illustrates this scheme, in which we assume that ten steps will give sufficient precision. We start at $(1/r', 0)$, where $1/r' = \cos h_1 \cdot \cos h_2 \cdot \cos h_3 \cdot \cdots \cdot \cos h_{10} = .637$.

The values of $\tan h_i$ for $i = 1, 2, \ldots$ can be precomputed, where $h_i = \pi/4, \pi/8, \pi/16, \ldots$, to mirror the chordic method. Each stage of the tangential algorithm would then require one decision—whether to use $+\tan h_i$ or $-\tan h_i$ and, through parametric equations (3), two multiplications and two additions or subtractions. Of these, the multiplications are the most problematic.

Multiplication is inherently more complicated than addition. Electronic devices multiply by repeated addition. Exactly how the repeated addition is implemented varies a great deal. At one extreme, simple and inexpensive microcontroller chips only have hardware for addition (Verle

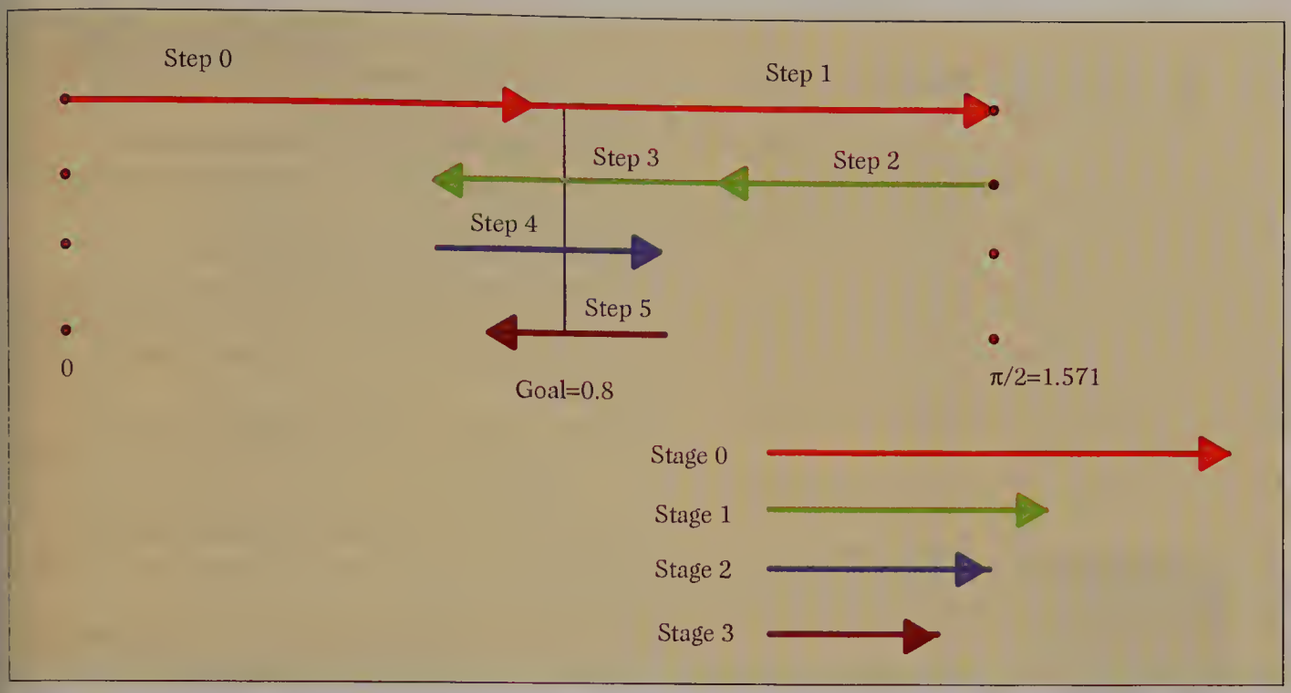

Fig. 8 The CORDIC convergence is shown for steps 0 through 3.

2008). The repeated additions must be explicitly programmed. At the other extreme, computers often use more expensive chips with built-in hardware to accomplish the repeated additions efficiently so that multiplication is nearly as quick as addition. Turley (2003) demonstrates how widely the performance can vary between chips. Whether measured in time or money, multiplication can be expensive.

Calculators cost less than computers in part because they use simpler chips. Many calculators use the CORDIC algorithm. CORDIC refines the tangential algorithm described above to eliminate the explicit multiplications by using a clever choice for the values of the angle h.

THE CORDIC ALGORITHM

Volder (1959) invented the CORDIC (COordinate Rotation DIgital Computer) method for real-time position calculations in aviation. Schelin (1983) proved that the method converged. Muller (1997) presented extensions of the method that compute logarithms and exponential functions.

In the CORDIC algorithm, the values of h are chosen depending on the format in which the calculator or computer stores numbers. For simplicity we will assume that numbers are stored in binary notation. If the calculator uses some other scheme, such as binary coded decimal (BCD), then a different set of angles will be used.

As stated before, a number in binary format can be easily multiplied by any power of 2 by shifting the binary point. Thus, we choose $h_i = \arctan(1/2^i)$, so that at the ith stage of the CORDIC algorithm, in equations (3), we are multiplying by $t_i = \pm 1/2^i$. Because we multiply by shifting, the CORDIC algorithm is called a "shift and add" algorithm.

This technique of reducing multiplications to shifts comes at a price. We are no longer mirroring the bisection scheme of the chordic algorithm. For instance, in **figure 5**, we must take care that the desired angle is between angle x and angle $y = x - h$. We do so by repeating the move through angle $h = \arctan(2^{-k})$ at each stage until we pass the desired angle. Thus, each stage in which $h = \arctan(2^{-k})$ may require more than one step in which we move through the same angle h. In contrast, the chordic algorithm requires only one step at each stage where $h = \pi/4, \pi/8, \pi/16, \ldots$.

For example, to get from step 2 of the table shown in **figure 7** to step 3, we use equations (3) with $P_2 = (0, 2)$ and $t = -1/2$ to get the coordinates of P_3 as $(0 - 2 \cdot (-1/2), 2 + 0 \cdot (-1/2)) = (1, 2)$, and we have found the new (p, q) coordinates through a "shift and add." We find $angle_3 = angle_2 - 0.464 = 1.5171 - 0.464 = 1.107$ using the precomputed value of $\arctan(1/2) = 0.464$.

Because the desired angle, 0.8, is not between 1.5171 and 1.107, we remain at stage 1 and again use $t = -1/2$ to find $P_4 = (1 - 2 \cdot (-1/2), 2 + 1 \cdot (-1/2)) = (2, 1.5)$ and $angle_4 = angle_3 - 0.464 = 1.107 - 0.464 = 0.643$. (**Fig. 7** shows 0.644 because it is rounding an Excel computation with more than three decimal digits.) The desired angle, 0.8, is now bracketed between 1.107 and 0.643, so we can move to stage 2 on the next step and use $t = 1/4$. We change the sign of t because we passed the desired angle on the last step of the previous stage and want to move back toward the desired angle. **Figure 8** illustrates this convergence process for stages 0 through 3 of the CORDIC computation given in **figure 7**.

stage	old angles	average	new angles	old sin	old cos	averages	new sin	new cos
0	0.000	0.785	0.785	0.000	1.000	0.500	0.707	0.707
	1.571		1.571	1.000	0.000	0.500		
1	0.785	1.178	0.785	0.707	0.707	0.854	0.924	0.383
	1.571		1.178	1.000	0.000	0.354		
2	0.785	0.982	0.785	0.707	0.707	0.885	0.831	0.556
	1.178		0.982	0.924	0.383	0.545		
3	0.785	0.884	0.785	0.707	0.707	0.769	0.773	0.634
	0.982		0.884	0.831	0.556	0.631		
4	0.785	0.834	0.785	0.707	0.707	0.740	0.741	0.672
	0.884		0.834	0.773	0.634	0.671		
5	0.785	0.810	0.785	0.707	0.707	0.724	0.724	0.690
	0.834		0.810	0.741	0.672	0.689		
6	0.785	0.798	0.798	0.707	0.707	0.716	0.716	0.698
	0.810		0.810	0.724	0.690	0.698		
7	0.798	0.804	0.798	0.716	0.698	0.720	0.720	0.694
	0.810		0.804	0.724	0.690	0.694		
8	0.798	0.801	0.798	0.716	0.698	0.718	0.718	0.696
	0.804		0.801	0.720	0.694	0.696		
9	0.798	0.799	0.799	0.716	0.698	0.717	0.717	0.697
	0.801		0.801	0.718	0.696	0.697		
10	0.799	0.800	0.800	0.717	0.697	0.717	0.717	0.697
	0.801		0.801	0.718	0.696	0.697		

Fig. 9 This sample chordic approximation of the sine and cosine of an angle of 0.8 radians arrives at the respective results of .717 and .697 after ten steps.

On the last step at each stage—say, stage i—we pass the desired angle, which is bracketed by two angles that differ by $\arctan(2^{-i})$. Hence, the error in the sine and cosine after that last step does not exceed $\arctan(2^{-i})$. Because the values of $\arctan(2^{-i})$ decrease to zero, the sine and cosine produced by this version of the CORDIC method converge.

Figure 9 shows the computations for the chordic method corresponding to the tangential method given in **figure 6**. The chordic and corresponding tangential algorithms reach three-place accuracy after nine stages of one step each, well before the guaranteed value of twelve steps. The CORDIC algorithm (see **fig. 7**) requires fifteen total steps in ten stages for the same accuracy.

AFTERTHOUGHTS

I have often taught the basic chordic method (without improvements), to my precalculus classes. I have not used CORDIC in the classroom. However, the approach given here should be suitable for undergraduates in a numerical analysis class.

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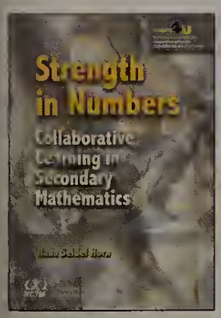
Strength in Numbers: Collaborative Learning in Secondary Mathematics, Ilana Seidel Horn, 2012. Grades 6–12, Higher Education. 100 pp., $29.95 paper. ISBN 978-0-87353-663-9. Stock no. 13791. National Council of Teachers of Mathematics; www.nctm.org.



This book does an excellent job of tackling the complexity of issues affecting collaborative learning environments. Its greatest strength is that it offers teachers an array of research-based knowledge in an accessible way to equip them to attend to the many considerations involved in implementing collaborative learning.

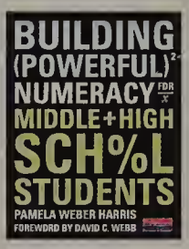
Strength in Numbers begins by comparing opportunities afforded by traditional and collaborative learning environments, a discussion that would be helpful to teachers still wondering about the benefits of collaborative learning. The book addresses the literature relevant to many matters that influence collaborative learning environments: equity, developing classroom norms, monitoring status, selecting group-worthy tasks, developing student interdependence, and the teacher's role during implementation. In each section, the author introduces practical research-based strategies for teachers to apply without oversimplifying the subjects.

The wide array of intertwined issues that this book covers could be a lot for a novice to process initially. Ongoing engagement with the text while attempting the strategies proposed would be a productive way to use this book. Whether a teacher is just beginning to consider collaborative learning or has experience with it, *Strength in Numbers* offers strategies to help critique the current learning environment and identify ways to improve it.

—Lindsay Keazer
Michigan State University
East Lansing, MI

FROM OTHER PUBLISHERS

Building (Powerful) Numeracy for Middle + High School Students, Pamela Weber Harris, 2011. 184 pp., $26.25 paper. ISBN 978-0-325-02662-6. Heinemann; www.heinemann.com.



This book describes effective ways to strengthen students' number and operational sense through the use of purposely chosen mathematical models and strings of mental mathematics problems. Harris shows how such tools provide the foundation for higher-level mathematics. For example, using number strings helps students understand the dual meanings of subtraction as both the removal and the difference and thus provides a foundation for understanding the slope of a line, integer addition and subtraction, the distance formula, and differential and integral calculus. Moreover, this book presents middle school and secondary school teachers with the computational strategies and models now being taught in some elementary school classrooms.

Building Powerful Numeracy for Middle + High School Students is a valuable resource for middle and secondary school teachers. However, I was surprised to discover several minor errors affecting the mathematics sprinkled throughout a book focusing on numeracy. An example of one typographical error follows: Harris describes why and how teachers should capitalize on students' experiences with money to help them make sense of fractions and decimals. She explains how students can find 120 by using reasoning about the quotient of a dollar divided by 20. She then notes: "Since there are 20 nickels in a dollar, $1.00 ÷ 10 = $0.05" (p. 109).

I recommend this book for both teachers and teacher educators. I plan to use it in my own courses for preservice teachers.

—Kelly McCormick
University of Southern Maine
Portland, ME

Prices of software, books, and materials are subject to change. Consult the suppliers for the current prices. The comments reflect the reviewers' opinions and do not imply endorsement by the National Council of Teachers of Mathematics.

MY FAVORITE lesson

Sonali Raje, Gail Kaplan, and Michael Krach

The Chinese Number Pyramid Puzzle

Puzzles are excellent tools for promoting logical reasoning, observational skills, and critical thinking abilities among students. The Chinese Number Pyramid is one such puzzle that requires students to observe data, analyze them critically, and obtain a logical solution. We have used this puzzle as part of a series of STEM workshops for high school students. The mathematics portion was designed to develop interest in problem-solving and critical thinking skills in preparation for college. The following narrative shows actual observations during the activity. We find that the puzzle engages students in a challenging, thought-provoking experience that enables them to develop an interest in mathematical problem solving.

The Chinese Number Pyramid Puzzle states: Discover the pattern in the following series of numbers and complete the succeeding lines, keeping in mind that each line of the sequence is uniquely determined:

1
11
21
1211
111221

To promote mathematical discourse, students are encouraged to work in pairs. Typically they start by adding the numbers in each row. Several pairs hypothesize that the sum of the numbers in each row is the Fibonacci sequence: 1, 2, 3, 5, and 8, …, in which each number is the sum of the two previous numbers.

By this logic, the sum of the next row would be 13. One group of students suggests a random permutation of 1s and 2s that add to 13. Because there are numerous permutations of 1s and 2s that add up to 13, the next line is not uniquely determined, so this educated guess is incorrect. Students continue to discuss potential solutions with peers. At this point, a facilitator shares that the only skills required for solving the problem are learned in first grade: reading and counting. That remark helps motivate students.

After working on the problem for about twenty minutes, the students ask for more help. Instead of providing the solution, the teacher provides the next element of the sequence: 312211.

When students notice the number 3, they immediately exclaim: "You didn't tell us we could use other numbers." The facilitators respond that no one said they couldn't. After about ten more minutes, at least one group finds the solution. As students in each group find the answer, they are asked to add another line to the solution. Often those who have been successful provide more and more leading hints, illustrating student-centered problem solving at its best. After a reasonable time, one student is asked to share her solution.

This deceptively challenging puzzle requires students to observe, critically analyze the given information, and use logical reasoning. It serves as an ideal model for encouraging students to think creatively and critically in mathematics.

SOLUTION (SPOILER ALERT!)

From left to right, read each number in the preceding line and count the number of times that number is present, consecutively. For example, the line 312211 begins with one 3 (1 3), which is followed by one 1 (1 1), two 2s (2 2), and then two 1s (2 1). Translating this sentence into numbers provides the next line: 13112221.

Note: We have named this puzzle in honor of a group of visiting Chinese educators who first shared this puzzle at our campus during a workshop on critical thinking and problem solving.

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The Back Page provides a forum for readers to share a favorite lesson. Lessons to be considered for publication should be submitted to mt.msubmit.net. Lessons should not exceed 600 words and are subject to abridgment.

Edited by ***Jennifer Wexler,*** *wexlerj@newtrier.k12.il.us, New Trier High School, Winnetka, IL*

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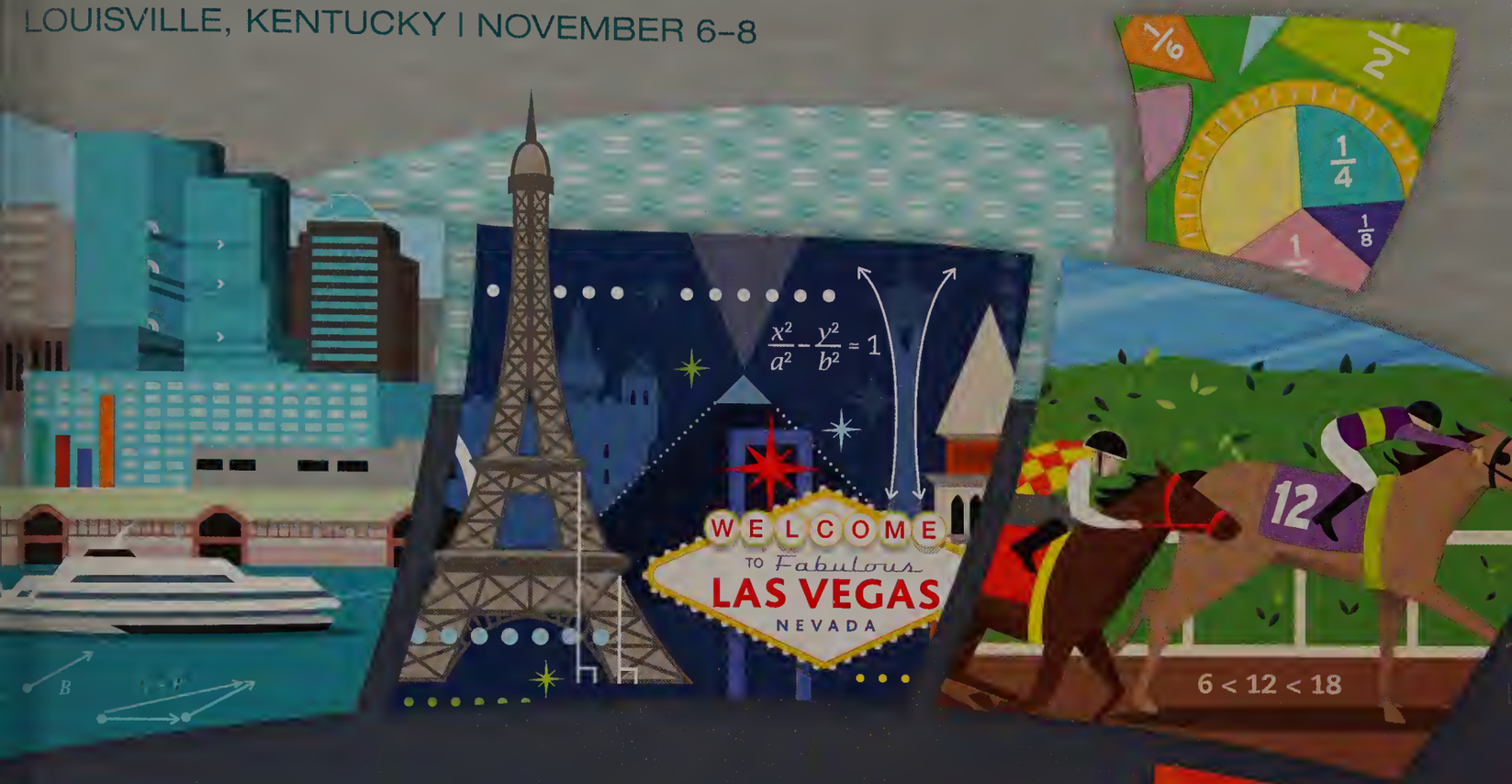

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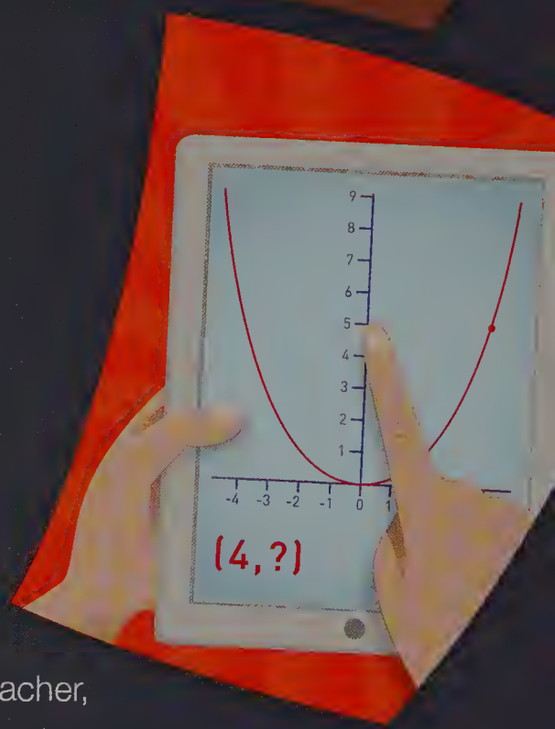

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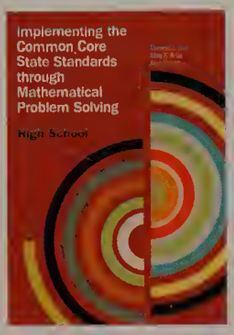

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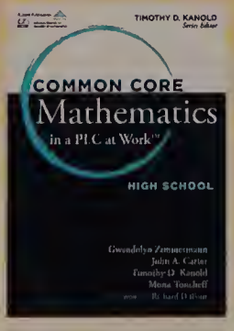

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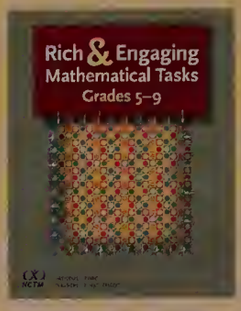

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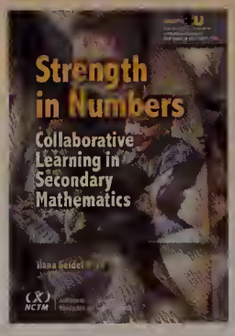

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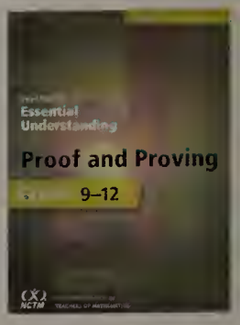

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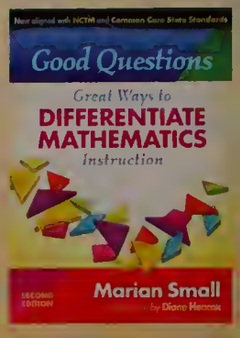

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